Henkel C, Sauer T-O, Proukakis NP.

Cross-over to quasi-condensation: mean-field theories and beyond.

Journal of Physics B 2017, 50(11), 114002.

DOI link to article:

https://doi.org/10.1088/1361-6455/aa6888

Date deposited:

16/05/2017

Cross-over to quasi-condensation: mean-field theories and beyond

2017 J. Phys. B: At. Mol. Opt. Phys. 50 114002

(http://iopscience.iop.org/0953-4075/50/11/114002)

OPEN ACCESS
IOP Publishing
Journal of Physics B: Atomic, Molecular and Optical Physics
J. Phys. B: At. Mol. Opt. Phys. **50** (2017) 114002 (19pp)
https://doi.org/10.1088/1361-6455/aa6888

Cross-over to quasi-condensation: mean-field theories and beyond

Carsten Henkel[1], Tim-O Sauer[1,2] and N P Proukakis[2,3]

[1] Institute of Physics and Astronomy, University of Potsdam, Karl-Liebknecht-Str. 24/25, 14476 Potsdam, Germany
[2] Joint Quantum Centre (JQC) Durham-Newcastle, School of Mathematics and Statistics, Newcastle University, Newcastle upon Tyne NE1 7RU, United Kingdom

E-mail: nikolaos.proukakis@ncl.ac.uk

Received 13 January 2017, revised 17 March 2017
Accepted for publication 22 March 2017
Published 15 May 2017

Abstract
We analyze the cross-over of a homogeneous, weakly interacting Bose gas in one dimension from the ideal gas into the dense quasi-condensate phase. We review a number of mean-field theories, perturbative or self-consistent, and provide accurate evaluations of equation of state, density fluctuations, and correlation functions. A smooth crossover is reproduced by classical-field simulations based on the stochastic Gross–Pitaevskii equation and the Yang–Yang solution to the one-dimensional Bose gas.

Keywords: quantum gases, Bose–Einstein condensation, phase transition, critical fluctuations

(Some figures may appear in colour only in the online journal)

The achievement of Bose–Einstein condensation in ultracold, dilute atomic vapors has opened a wide research field at the crossroads of quantum optics and condensed-matter physics [1, 2]. It was clear from the beginning that the scenarios of the ideal Bose gas, well-known from thermodynamics, are not sufficient because of interactions between the atoms. In contrast to liquid helium where interactions are strong [3, 4], ultracold vapors can be modeled nearly from first principles, the s-wave scattering length being the main relevant coupling constant. In lower spatial dimensions, interactions qualitatively change the phase diagram and lead to the emergence of a paired vortex phase (Kosterlitz–Thouless transition, 2D) or a quasi-condensate (1D). The phase boundaries are fuzzy, however, and the cross-over region, as it has been called, poses a challenge to conventional pictures. Indeed, thermal and quantum fluctuations that are prominent anyway in lower dimensions have to be modeled in the presence of interactions. As the density is lowered through the cross-over region, density fluctuations become comparable to phase fluctuations, so that a Luttinger liquid approach [5] breaks down. It is then questionable whether one can operate a clean splitting into a 'quasi-condensate' and a 'thermal cloud' familiar from spontaneous symmetry breaking. This may explain why the cross-over is so difficult to describe with mean-field theories that build on the Bogoliubov prescription.

In this paper, we provide a critical assessment of mean-field theories for the description of the cross-over in a homogeneous, weakly interacting, one-dimensional Bose gas between the ideal gas and the quasi-condensate. These approaches have the common feature that the many-body system is broken down to relevant collective observables that are treated as 'hydrodynamic fields', examples being the (total) density or a c-number valued condensate field. The hydrodynamic fields parametrize an approximate form of the many-body Hamiltonian which is simple enough to be diagonalised in a quasi-particle basis with a well-defined dispersion relation. This permits to compute different mean values and correlation functions. Throughout the paper, we exclude the case of strong interactions which leads to fermionisation (impenetrable bosons, Tonks–Girardeau regime). Most of the theories considered here gradually break down as this regime is approached.

[3] Author to whom any correspondence should be addressed.

There are a number of variants for mean-field theories: some are based on perturbative expansions (weak interactions, weak density fluctuations) whose validity becomes doubtful in the cross-over region, others are constructed in a 'self-consistent' way and may suggest a comprehensive treatment of both regimes. We give an overview on several approaches and work out in detail the equation of state, density fluctuations, and correlation functions. There are numerous approaches that have been implemented to improve on the simple mean-field theories. For a unified-notation review, the readers are referred to [2, 6]. The '$G1$' variant of the Hartree–Fock–Bogoliubov approximation attempts to fix the issue of a gapless dispersion relation by carefully observing features a successful theory might have—indeed it had some success in modeling experiments [7, 8]. Much has been written about a gapless spectrum in relation to the Goldstone and Hugenholtz–Pines theorems [9, 10]. We find here that its impact is marginal with respect to the performance of a mean-field theory in the cross-over of the one-dimensional Bose gas. In fact, we analyze two approximations, one gapless (modified Popov approximation of [11–13]), the other not (Hartree–Fock–Bogoliubov theory developed by Walser and the Holland group [14–16]). Their predictions for the equation of state and density fluctuations are qualitatively similar, however, as discussed later in the paper. They also fail both in providing a smooth description as the chemical potential crosses zero, and predict a critical point (discontinuity in the equation of state). This artefact has been noted before for mean-field theories in three dimensions, see [17, 18]. A different fate arises when the self-consistent and gapless theory of Yukalov and Yukalova [17, 19] is extrapolated to a one-dimensional system. The integrals giving the non-condensate density and other quantities diverge in the infrared (IR), similar to the simpler Bogoliubov theory. In [17], this is claimed to be removed with dimensional regularization, effectively subtracting the divergent piece, although the resulting 'density' becomes negative. An IR regularization has also been operated in the modified Popov theory, but following a different argument: the IR-divergent pieces were identified as spurious phase fluctuations and eliminated. The resulting expressions are discussed here.

Another important development was the construction of an expansion for large particle numbers, but in a number-conserving way, following arguments laid out in [20] and extended in [21–24]. We mention that for our system of interest, the homogeneous Bose gas in the thermodynamic limit, the predictions of mean-field theory are qualitatively quite similar, whether it is formulated in a number-conserving way or in the grand-canonical ensemble with symmetry breaking. We have checked this with the example of extended Bogoliubov theory developed by Mora and Castin [25]: one key technique, the projector orthogonal to the condensate mode, is irrelevant for a homogeneous system where the elementary excitations naturally appear at finite momentum. The expansions behind these approaches, for example in the fraction of non-condensed particles, are bound to break down in the cross-over because there is no condensate in the dilute phase. In the case of extended Bogoliubov theory, it is the assumption of weak density fluctuations that fails.

It turns out that none of the mean-field theories analyzed here describes the cross-over of the Bose gas from dilute to dense in a satisfactory way: some theories are simply restricted 'by construction' to either side of the phase boundary. Other theories give wrong predictions for one side, or suggest a critical point in the equation of state. Fortunately, it is possible to follow the cross-over with the help of complex-field simulations (stochastic Gross–Pitaevskii equation, sGP, for a review, see [6, 26, 27]). Proposals of this technique date back to Stoof's group [28, 29], Davis and the Burnett group [30], and Gardiner's group [31, 32]. See also related classical field work by Goral and the Rzążewski group [33]. The sGP has been applied to one-dimensional Bose correlations by one of us [13]. With a suitably chosen cutoff, its predictions are in excellent agreement with experiments in one [34, 35] and two dimensions [36, 37]. We find that these simulations successfully achieve a reasonable modeling of the entire cross-over. Another 'benchmark' is provided by the exact solution of the Lieb–Liniger model at finite temperature by Yang and Yang [38, 39]. This approach has been used to cross-check perturbative calculations of density fluctuations in the dilute phase by Kheruntsyan and the Shlyapnikov group [40, 41]. It also remains valid in the strongly interacting regime and may thus be used to quantify how weak interactions have to be in order for the other theories to provide a reasonable description of the physics.

To conclude with a comparison to experiments, it should be noted that many setups require modeling beyond the one-dimensional regime, mainly because the transverse confinement is not strong enough. As the ratio between trap frequencies is changed, one observes a 'dimensional cross-over' from a true three dimensional condensate to a one-dimensional quasi-condensate with large phase fluctuations [42]. Following relatively early anisotropic expansion experiments [43, 44], theoretical work on this has been performed by Al Khawaja *et al* [45] and Gerbier [46]. Experimental work by the Bouchoule group [47–50] demonstrated, for example, the breakdown of Hartree–Fock mean field theory by analyzing the density fluctuations, and mapped out the dimensional cross-over (for a review, see [51]). Setups deeply in the one-dimensional regime have been reported in [52, 53] where Yang–Yang thermodynamics could be checked. A recent experiment of the Pan group has also compared Yang–Yang theory to the equation of state in the cross-over [54]. The failure of mean field theories becomes manifest experimentally in the boundary regions of a trapped system [48]. For the comparison with theory, the local density approximation (LDA) is often applied. We check the accuracy of this approximation using sGP simulations for both a homogeneous system and a trapped one.

Structure of the paper: the problem setting and a few salient parameters are outlined in section 1. We discuss mean-field theories that do not operate a splitting of the Bose

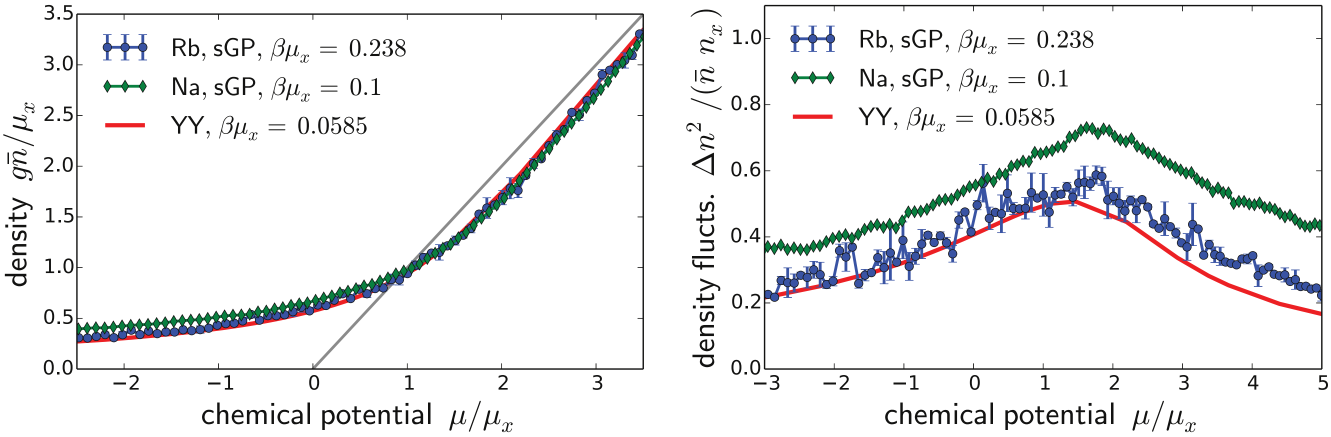

Figure 1. Illustration of the universal features of the dilute-to-dense cross-over for different atoms and temperatures, when scaled to cross-over units. Symbols (sGP): stochastic Gross–Pitaevskii equation using the parameters of table 2 [27], courtesy of Stuart Cockburn for the Rb data. Solid line (YY): numerical evaluation of the Yang–Yang solution to the finite-temperature Lieb–Liniger model [38, 39], courtesy of Karen Kheruntsyan. The temperature parameter for the YY data is $k_B T\hbar^2/(Mg^2) = 5 \times 10^3 = (\beta\mu_x)^{-3}$. The Lieb–Liniger parameter can be written as $\gamma = (\beta\mu_x)^2(g\bar{n}/\mu_x)^{-1}$ and is small throughout our range of μ (weak interactions). (See also table 2.) (Left) Equation of state. (Right) Density fluctuations, expressed by the 'Mandel parameter' $\Delta n^2/\bar{n}$. (The name is chosen by analogy to super-Poissonian photon number distributions in laser theory.)

gas in components (section 2): the ideal gas and Hartree–Fock theory are covered [1]. The Bogoliubov approximation in section 3.1 introduces the condensate concept, although it suffers from serious IR divergencies in low dimensions. An extended version that applies to a quasi-condensed gas whose density fluctuations are weak has been developed by Mora and Castin [25], section 3.2. So-called self-consistent theories are covered in section 4, beginning with the modified PJPBopov theory, section 4.1. This is based on suitably regularized expressions for the non-quasi-condensate component. We also illustrate in this section the many-body effects that renormalise the interatomic scattering properties. The last mean-field theory is a variant of Hartree–Fock–Bogoliubov developed by Walser, section 4.2. The results are discussed in section 5 and compared to stochastic simulations with the Gross–Pitaevskii equation. The appendices summarize more technical material related to high- and low-temperature approximations and to numerical aspects.

1. Problem setting

We consider a gas of N bosonic particles of mass M, strongly confined into a one-dimensional trap of length L, and in thermal equilibrium at temperature T. Throughout we work in the thermodynamic limit of a large system with density $\bar{n} = N/L$. This density is controlled by the chemical potential μ, and the interaction energy per particle is given by gn with a positive constant g. In the language of second quantization, the Hamiltonian H is

$$H = \int \mathrm{d}z \left[\frac{\hbar^2}{2M} \frac{\mathrm{d}\psi^\dagger}{\mathrm{d}z} \frac{\mathrm{d}\psi}{\mathrm{d}z} + \frac{g}{2} \psi^\dagger \psi^\dagger \psi \psi - \mu \psi^\dagger \psi \right], \tag{1}$$

where the field satisfies the bosonic commutation relations $[\psi(z), \psi^\dagger(z')] = \delta(z - z')$. A list of relevant observables is given in table 1. Note that for the homogeneous system we consider in this paper, local averages like the mean density $\bar{n}$ are spatially constant, while correlation functions depend only on the distance $z - z'$ between the observation points.

The characteristic scales for the cross-over can be motivated as follows. For negative chemical potentials, the density is low, and the ideal gas is a good approximation. The statistics of the complex field operator ψ is then Gaussian, and from its fourth moment, one finds that density fluctuations are significant: $\langle n^2 \rangle \approx 2\bar{n}^2$. The cross-over is reached from below when the interaction energy in equation (1) becomes relevant, i.e., for $\mu \sim -g\bar{n}$. When the density is estimated with the degenerate ideal gas formula (first term of equation (8) below), we get $\mu \sim -\mu_x$ with characteristic energy and density scales

$$\mu_x = \left(\frac{g M^{1/2} k_B T}{\hbar} \right)^{2/3}, \qquad n_x = \mu_x / g. \tag{2}$$

We shall see below that μ_x gives the typical width of the cross-over region around $\mu = 0$. Repulsive interactions stabilize the gas so that also positive chemical potentials become accessible. On this dense side of the cross-over, density fluctuations get weaker: $\langle n^2 \rangle \approx \bar{n}^2$. The phase still fluctuates strongly enough to prevent the formation of long-range order, leading to the quasi-condensate concept.

Typical numbers are listed in table 2 for two different atoms [55, 56]. We use the standard formula $g = 2\hbar\omega_\perp a_s$ for the one-dimensional interaction constant, assuming that the transverse confinement gives the highest energy scale. We define the thermal wavelength as $\lambda = \hbar\,(M k_B T)^{-1/2}$ and the healing length for a given density n as $\xi = \hbar\,(4Mgn)^{-1/2}$ (see table 4). In the cross-over, the two length scales are comparable, while the density is still high enough to be far from the Tonks–Girardeau limit. This can be expressed in terms of the Lieb–Liniger parameter $\gamma = Mg/(\hbar^2 \bar{n}) = (2 n_x \xi_x)^{-2}\, n_x/\bar{n} \ll 1$ [38].

An illustration of the relevance of the energy scale μ_x (equation (2)) is provided by figure 1 where we show the

Table 1. Hydrodynamic fields. The colons denote normal ordering of the field operators.

(Quasi)condensate	$\phi = \langle\psi\rangle,\ n_q = \|\phi\|^2$
Mean density	$\bar{n} = \langle n\rangle = \langle\psi^\dagger\psi\rangle$
Field correlations	$G_1(z - z') = \langle\psi^\dagger(z)\psi(z')\rangle$
Density correlations	$C(z - z') = \langle n(z)n(z')\rangle - \bar{n}^2$
Pair correlations	$G_2(z - z') = \langle: n(z)n(z'):\rangle$
Thermal density	$n' = \bar{n} - n_q = G_1(0) - n_q$
Anomalous average	$m' = \langle\psi^2\rangle - \phi^2$

equation of state and the normalized density fluctuations for two 'benchmark theories': the first is based on numerical simulations of the sGP equation [6], the second is the *ab initio* solution of the finite-temperature Lieb–Liniger model (Yang–Yang thermodynamics) [39]. We parametrize the data by the dimensionless inverse temperature $\beta\mu_x = \mu_x/k_B T$. Since this scales as $\sim T^{-1/3}$, the temperature range covers nearly two orders of magnitude, but the data 'collapse' into a quite narrow band. The scale μ_x obviously captures the width of the cross-over zone versus μ with excellent accuracy. The density fluctuations (right panel) show a slightly larger scatter, but this is due in part to a dependence on the numerical parameters like spatial grid spacing. We emphasize that with an appropriate choice of sGP simulation parameters, excellent agreement with experimental data has been found, see [34–37]. Equilibrium results are essentially sensitive only on one parameter, the cutoff energy related, e.g., to the spatial grid of the simulation; it can be fixed by the requirement of reproducing the total atom number (see [26] for an extensive discussion).

An additional reason for the larger scatter comes from experiment: it is known that thermometry can be quite challenging based on the equation of state alone (illustrated by the collapse of the simulation data). Density fluctuations provide a more sensitive access to temperature [50], illustrated here by the deviations between curves at different temperatures.

2. One-component theories

2.1. Ideal gas

The simplest example is the ideal gas ($g = 0$) where the Hamiltonian is bilinear and diagonal in the plane wave basis (dispersion relation $\epsilon(k) = \hbar^2k^2/2M, -\infty < k < +\infty$)

$$H = \int \mathrm{d}k\,(\epsilon(k) - \mu)a^\dagger(k)a(k),$$
$$\psi(z) = \int \mathrm{d}k\ a(k)\frac{\exp(\mathrm{i}kz)}{\sqrt{2\pi}} \tag{3}$$

with annihilation and creation operators $[a(k), a^\dagger(k')] = \delta(k - k')$. In thermal equilibrium, we have $\langle a(k)\rangle = 0$ and

Table 2. Two sets of typical parameters used in simulations (stochastic Gross–Pitaevskii equation).

	Na-23	Rb-87
Scattering length a_s	$51.97\ a_0$	$95.41\ a_0$
Transv. confinement $\omega_\perp$	1.46 kHz	4 kHz
Interaction g	0.39 nK μm	1.938 nK μm
Temperature T	7 nK	50 nK
Thermal wavelength λ	1.74 μm	0.33 μm
$\hbar^2 k_B T/Mg^2$	10^3	74
Cross-over chem. pot. μ_x	0.70 nK (14.6 Hz)	12 nK (250 Hz)
Cross-over density n_x	1.8 μm^{-1}	6.1 μm^{-1}
Healing length ξ_x	2.75 μm	0.34 μm
Lieb–Liniger $\gamma(n_x)$	0.010	0.057

recover Bose–Einstein statistics ($1/\beta = k_B T$)

$$\langle a^\dagger(k)a(k')\rangle = N(\epsilon(k) - \mu)\delta(k - k')$$
$$N(\epsilon(k) - \mu) = \frac{1}{\exp[\beta(\epsilon(k) - \mu)] - 1}. \tag{4}$$

The field correlation function is denoted $G_1(z - z') = \langle\psi^\dagger(z)\psi(z')\rangle$ and given by

$$G_1(x) = \int\frac{\mathrm{d}k}{2\pi}N(\epsilon(k) - \mu)\exp(\mathrm{i}kx). \tag{5}$$

The 'Boltzmann approximation' $N(\epsilon - \mu) \approx \mathrm{e}^{-\beta(\epsilon-\mu)}$ applies in the regime $\epsilon - \mu \gg k_B T$ and gives a Gaussian correlation function

$$G_1(x) \approx \frac{\mathrm{e}^{\beta\mu}}{\sqrt{2\pi}\ \lambda}\mathrm{e}^{-x^2/(2\lambda^2)} \tag{6}$$

with a correlation length set by the thermal wavelength λ. If large distances are of interest or the chemical potential approaches the critical value $\mu = 0$, a different approximation is required. The Rayleigh–Jeans approximation, $N(\epsilon - \mu) \approx k_B T/(\epsilon - \mu)$, captures the contribution of small-k modes with high degeneracy and yields an exponential shape

$$G_1(x) \approx \frac{\ell}{\lambda^2}\mathrm{e}^{-|x|/\ell} \tag{7}$$

with a much larger correlation length $\ell = \hbar(-2M\mu)^{-1/2}$ (see table 4). We sketch in appendix A.1 an expansion that gives the next-to-leading order corrections for $\lambda \ll \ell$:

$$\bar{n}(\mu) \approx \frac{\ell}{\lambda^2} - \frac{a_1}{\lambda} + \frac{a_2}{2}\frac{\lambda}{\ell^2}. \tag{8}$$

They are relatively important and involve the positive coefficients $a_1 = -\zeta\left(\frac{1}{2}\right)\Big/\sqrt{2\pi} \approx 0.5826$ and $a_2 = -\zeta\left(-\frac{1}{2}\right)\Big/\sqrt{2\pi} \approx 0.0830$ and the regularized zeta function.

The density correlation function (table 1) is computed from the Wick theorem because the Hamiltonian H (equation (3)) generates Gaussian statistics. This results in

$$C(x) = \bar{n}\,\delta(x) + |G_1(x)|^2, \tag{9}$$

where the first term represents 'shot noise' (it arises from putting the field operators into normal order). The second term is called 'bunching' and increases the density fluctuations to the level $\langle:n^2:\rangle = 2\bar{n}^2$.

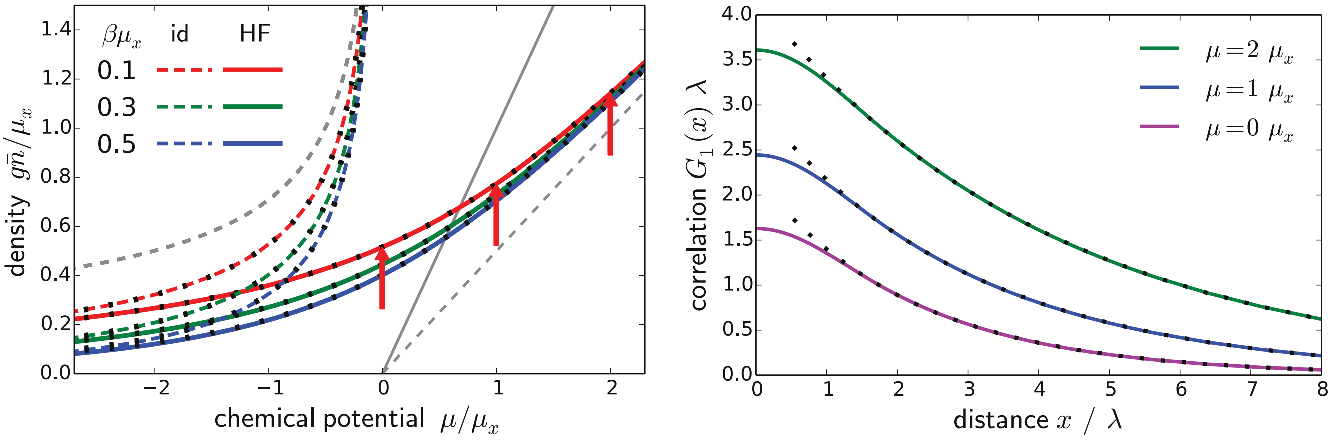

Figure 2. Comparison of one-component mean-field theories. (Left) Equation of state $\bar{n}(\mu)$. Dashed: ideal gas, solid: Hartree–Fock theory. Dashed gray curve: classical approximation (first term of equation (8)). Dotted lines, superimposed: low-energy approximations based on equation (8). Straight solid line: pure condensate $\mu = g\bar{n}$, straight dashed: Hartree–Fock asymptote $\mu = 2g\bar{n}$. Chemical potential and density scaled to cross-over units (equation (2)). Arrows: values of μ selected for right panel. (Right) Correlation function $G_1(x)$ for three different densities (marked by red arrows on the left), Hartree–Fock theory. Temperature such that $\beta\mu_x = 0.1$. Dotted curves: low-energy approximation based on equation (7); their characteristic length (decay to $1/e$) is $\ell \approx \bar{n}\lambda^2$. The same results would have been obtained for an ideal gas at the chemical potentials (from top to bottom) $\mu \approx -0.285, -0.547, -1.03\ \mu_x$ (see table 3).

2.2. Interacting gas: Hartree–Fock

Hartree–Fock theory is probably the oldest mean-field theory; it is treating the interactions in a Bose gas in terms of an additional potential (the 'mean field'). The Hamiltonian is approximated in the plane-wave basis by

$$H \approx \int \mathrm{d}k\,(\epsilon(k) + 2g\bar{n} - \mu)a^{\dagger}(k)a(k). \tag{10}$$

This shift of the chemical potential gives the same equation of motion as the full interaction Hamiltonian when correlation functions are factorized in a Gaussian approximation [9]. As long as $2g\bar{n} > \mu$, Bose–Einstein statistics can be applied as for the ideal gas, and we get the following implicit equation for the (mean) density

$$\bar{n} = \int \frac{\mathrm{d}k}{2\pi} \frac{1}{\exp[\beta(\epsilon(k) + 2g\bar{n} - \mu)] - 1}. \tag{11}$$

To work out this formula, we use an ideal-gas chemical potential $\mu_i < 0$ as parameter and plot $\bar{n}_{\mathrm{id}}(\mu_i)$ versus $\mu = \mu_i + 2g\bar{n}_{\mathrm{id}}(\mu_i)$. The approximation shown in equation (8) can also be used here; it is fairly accurate, as long as $\beta\mu_x \lesssim 0.1$ (see dotted lines in figure 2(left)).

In the leading order, we get the explicit expression

$$\mu \approx 2g\bar{n} - \frac{k_B T}{2(\bar{n}\lambda)^2} = 2g\left(\bar{n} - \frac{n_x^3}{4\bar{n}^2}\right). \tag{12}$$

Right at the cross-over $\mu = 0$, we have $\bar{n} = 2^{-2/3}\,n_x \approx 0.63\,n_x$ where the cross-over density scale $n_x = \mu_x/g$ is defined by equation (2). In the dense case ($\mu \gg \mu_x$), note again the collapse of the data in cross-over units over a wide range of temperatures. The equation of state $\mu \approx 2g\bar{n}$, however, is off by 50% compared to Bogoliubov theory (equation (14) below, see figure 2(left)). This will be improved by a more advanced mean-field theory. For ease of comparison between theories discussed in this paper, the equation of state and other expressions are shown together in table 3.

The correlation function $G_1(x)$ of Hartree–Fock theory is formally given by the same integral (5) as for the ideal gas, but evaluated at the self-consistent chemical potential, as shown in figure 2. The density correlations are given by equation (9) because of Gaussian statistics. They therefore show the same bunching as the ideal Bose gas. At large distances, they feature an exponential decay on a length scale $\ell \approx \bar{n}\lambda^2$ (see equations (7), (8)) that is much larger than the thermal wavelength. This behavior does not capture the strong differences between phase and density fluctuations that characterize the dense phase [5]. (An improved version of Hartree–Fock including the many-body renormalization of particle interactions is briefly discussed in section 4.1.3.)

3. Expansion around a (quasi) condensate

3.1. Bogoliubov theory

This mean field approach is very successful in three dimensions and implements the concept of spontaneous symmetry breaking in the dense phase. Although it is not directly applicable in lower dimensions, it provides an introduction to the key concepts. The basic idea is the so-called Bogoliubov shift where the field operator is split into a c-number valued field (the 'condensate') and fluctuations, $\psi \mapsto \phi + \hat{\psi}$. The Hamiltonian is expanded up to second order in the fluctuations, and the condensate is determined by the requirement that the terms linear in $\hat{\psi}$ vanish. This gives the Gross–Pitaevski equation

$$-\frac{\hbar^2}{2M}\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + g|\phi(z)|^2\phi = \mu\phi. \tag{13}$$

Table 3. Formulas for mean-field theories. The dispersion relations are given in the thermodynamic sense: the energy ε appears in the Bose–Einstein factor $N(\varepsilon)$. Non-condensate density, anomalous density. Equation of state. The lower limits of the chemical potential are taken from table 7.2 of [57], $a_1 = -\zeta(1/2)/\sqrt{2\pi} \approx 0.583$. This is based on a low-energy (high-temperature) expansion; higher-order corrections are $\mathcal{O}(\beta\mu_x)$ or $\mathcal{O}(\beta\mu_x)^{3/2}$.

Dispersion relation		
Ideal gas	$\epsilon - \mu \equiv \hbar^2k^2/2M - \mu$	(Gap)
Hartree–Fock	$\epsilon + 2gn' - \mu$	(Gap)
Bogoliubov	$E \equiv \sqrt{\epsilon\,(2gn_c + \epsilon)}$	
Mora–Castin	E (with $gn_c \mapsto \mu$)	
Modified Popov	E (with $n_c \mapsto n_q$)	
Walser	$\sqrt{(\epsilon - 2gm')(2gn_c + \epsilon)}$	(Gap)
Non-condensate density	$n' = \int\frac{dk}{2\pi}\ldots$	
Ideal gas	$N(\epsilon - \mu)$	
Hartree–Fock	$N(\epsilon + 2gn' - \mu)$	
Bogoliubov	$\frac{\epsilon + gn_c}{E}N(E) + \frac{\epsilon + gn_c - E}{2E}$	(IR divergent)
Mora–Castin	$\frac{\epsilon}{E}N(E) + \frac{\epsilon - E}{2E}$	(Non-positive)
Modified Popov	$\frac{\epsilon}{E}N(E) + \frac{\epsilon - E}{2E} + \frac{gn_q}{2(\epsilon + \mu)}$	
Walser	$\frac{\epsilon + g(n_c - m')}{E}N(E) + \frac{\epsilon + g(n_c - m') - E}{2E}$	
Anomalous density	$m' = \int\frac{dk}{2\pi}\ldots$	
Bogoliubov	$-\frac{gn_c}{E}\left(N(E) + \frac{1}{2}\right)$,	(IR divergent)
Walser	$-\frac{g(n_c + m')}{E}\left(N(E) + \frac{1}{2}\right)$	
Equation of state		
Ideal gas	$\bar{n} = n'_{\mathrm{id}}(\mu)$	$\mu < 0$
Hartree–Fock	$\bar{n} = n'_{\mathrm{id}}(\mu - 2g\bar{n})$	
Bogoliubov	$\mu = gn_c$	$\mu > 0$
Mora–Castin	$\mu = g\bar{n} + gn'$	$\mu/\mu_x \gtrsim 0.630 - a_1(\beta\mu_x)^{1/2}$
Modified Popov	$\mu = gn_q + 2gn'$	$\mu/\mu_x \gtrsim 1.89 - 2a_1(\beta\mu_x)^{1/2}$
Walser	$\mu = gn_c + 2gn' + gm'$	$\mu/\mu_x \gtrsim 2.11 - 2a_1(\beta\mu_x)^{1/2}$

For reasons of thermodynamic stability, one chooses the condensate with the largest possible density—which is spatially constant in a homogeneous system. We get the equation of state

$$\mu = g|\phi|^2 = gn_c \tag{14}$$

that leaves the phase of ϕ undetermined. The conventional choice of real and positive ϕ can be interpreted as a spontaneous breaking of the U(1)-symmetry of the field Hamiltonian (1). The self-interaction of the condensate contributes an energy density $\epsilon_c = -\frac{1}{2}gn_c^2$ to the Hamiltonian, corresponding to a pressure $p = gn_c^2/2$. These parameters allow for acoustic elementary excitations with a speed of sound c at long wavelengths set by $Mc^2 = \partial p/\partial n_c = gn_c$.

3.1.1. Quasi-particle spectrum. The part of the Hamiltonian that is of second order in $\hat{\psi}$ is diagonalized with the help of the Bogoliubov transformation

$$\hat{\psi}(z) = \int\frac{dk}{\sqrt{2\pi}}[b(k)u(k)\,e^{ikz} + b^\dagger(k)v(k)e^{-ikz}], \tag{15}$$

where the Bogoliubov amplitudes u and v are given in table 5. They are constrained by $|u|^2 - |v|^2 = 1$ to make the operators b and $b^\dagger$ bosonic [commutation relation $[b(k), b^\dagger(k')] = \delta(k - k')$, as after equation (3)]. We note that both u and v are proportional to the phase factor $e^{i\varphi}$ involving the condensate phase. The operator $b(k)$ annihilates a quasi-particle with energy $E(k)$ given by the Bogoliubov dispersion relation (table 5) where the acoustic branch involves the speed of sound $c = (gn_c/M)^{1/2}$ consistent with the hydrodynamic argument mentioned above. In this long wavelength limit, the Bogoliubov amplitudes become comparable and large, $u \sim -v \gg 1$.

Finally, the field Hamiltonian is truncated at second order in $\hat{\psi}$, taking the following form,

$$H \approx \epsilon_0 L + \int dk\; E(k)b^\dagger(k)b(k). \tag{16}$$

Table 4. Correlation lengths.

Ideal gas ($\mu < 0$)	
Correlation length	$\ell = \hbar\,(-2M\mu)^{-1/2} \approx \bar{n}\lambda^2$
Bogoliubov theory ($\mu > 0$)	
Phase diffusion length[a]	$\ell_\theta = n_c\lambda^2$
Healing length	$\xi = \hbar\,(4M\mu)^{-1/2}$
Extended Bogoliubov ($\mu > 0$)	
Phase correlation length	$\ell_\theta = 2\bar{n}\lambda^2$
Density correlation length	$\xi = \hbar\,(4M\mu)^{-1/2}$
Modified Popov	
Phase correlation length	$\ell_\theta = 2n_q\lambda^2$
Density correlation lengths	$\xi_q = \hbar\,(4Mgn_q)^{-1/2}$ and $\sqrt{2}\,\xi$
Hartree–Fock–Bogoliubov	
Field and density correlation lengths	$\xi_c = \hbar\,(4Mgn_c)^{-1/2}$
and	$\xi_m = \hbar\,(-4Mgm')^{-1/2}$

[a] Over the distance ℓ_θ, the phase quadrature Y (equation (25)) has diffused such that its variance is comparable to the condensate density $n_c = \mu/g$.

The zero-point energy density ϵ_0 arises by putting the Bogoliubov operators b, $b^\dagger$ into normal order. It is given by the integral

$$\epsilon_0 - \epsilon_c = -\int\frac{\mathrm{d}k}{2\pi}E(k)|v(k)|^2 = -\int\frac{\mathrm{d}k}{2\pi}\frac{\mu^2/2}{E(k) + \epsilon(k) + \mu} \tag{17}$$

which converges and can be computed analytically (appendix A.3)

$$\epsilon_0 = -\frac{\mu n_c}{2} - \frac{\mu}{3\pi\xi}. \tag{18}$$

The Bogoliubov correction is small and scales with the root of the Lieb–Liniger parameter [38] $\sqrt{\gamma} \approx 1/(2n_c\xi)$ where ξ is the healing length of table 4. More analytical results at zero temperature in one and higher dimensions can be found in [58, 59].

The key problem of Bogoliubov theory in one dimension is the IR divergence of the non-condensate density n'. The latter is defined as

$$n' = \langle\psi^\dagger\psi\rangle - |\phi|^2 = \langle\hat{\psi}^\dagger\,\hat{\psi}\rangle. \tag{19}$$

In thermal equilibrium with respect to the approximate Hamiltonian (16), the modes corresponding to the $b(k)$'s have an occupation $N(E(k))$ (see equation (4)). Using the expansion (15) of the field operator, the thermal density is given by the integral

$$n' = \int\frac{\mathrm{d}k}{2\pi}n'(k), \tag{20}$$

$$n'(k) = N(E(k))|u(k)|^2 + [N(E(k)) + 1]|v(k)|^2. \tag{21}$$

The zero temperature limit $N(E) \to 0$ gives the so-called depletion density that arises by the scattering of virtual particles out of the condensate. Its integral is divergent in the IR because the Bogoliubov amplitude scales $|v(k)|^2 \sim 1/k$ at long wavelengths. The temperature-dependent part shows an

Table 5. Bogoliubov quasi-particles ($\mu = g|\phi|^2 > 0$).

Free particle energy	$\epsilon(k) = \hbar^2k^2/2M$
Dispersion relation	$E(k) = \sqrt{\epsilon(k)(2\mu + \epsilon(k))}$
Bogoliubov amplitudes	$u(k) = \mathrm{e}^{\mathrm{i}\varphi}\cosh\frac{1}{2}\alpha(k)$
(Condensate phase φ)	$v(k) = -\mathrm{e}^{\mathrm{i}\varphi}\sinh\frac{1}{2}\alpha(k)$
	$\cosh\alpha(k) = \frac{\epsilon(k)+\mu}{E(k)}$
	$\sinh\alpha(k) = \frac{\mu}{E(k)}$

even stronger divergence $\sim T/k^2$ so that Bogoliubov theory is only useful as a conceptual framework. Here is an explicit expression for the non-condensate distribution in k-space that will re-surface later (see also table 3)

$$n'(k) = \frac{\mu + \epsilon(k)}{2E(k)}\coth\frac{\beta E(k)}{2} - \frac{1}{2}. \tag{22}$$

For completeness, we mention that in (symmetry-broken) Bogoliubov theory, also the so-called anomalous average $m = \langle\psi\psi\rangle = \phi^2 + m'$ is non-zero. In k-space, it involves the product of the two Bogoliubov amplitudes,

$$m'(k) = [2N(E(k)) + 1]u(k)v(k) \tag{23}$$

$$= -\frac{\mu}{2E(k)}\coth\frac{\beta E(k)}{2} \tag{24}$$

but its integral is also IR-divergent. Note that we fixed the condensate phase to $\varphi = 0$ in the second line.

3.1.2. Density and phase diffusion. Finite results within Bogoliubov theory can be produced by considering spatial increments, similar to Brownian motion. Consider the difference $\Delta\psi(z, z') = \psi(z) - \psi(z')$ and its real and imaginary parts, assuming real and positive ϕ. The average vanishes, $\langle\Delta\psi\rangle = \langle X + \mathrm{i}Y\rangle = 0$, and for the (co)variances, we find $\langle{:}XY + YX{:}\rangle = 0$ and the integral representations

$$\langle{:}X^2{:}\rangle = \int\frac{\mathrm{d}k}{2\pi}(1 - \cos kx)\left\{\frac{\epsilon}{2E}\coth\frac{\beta E}{2} - \frac{1}{2}\right\}$$
$$\langle{:}Y^2{:}\rangle = \int\frac{\mathrm{d}k}{2\pi}(1 - \cos kx)\left\{\frac{\epsilon + 2\mu}{2E}\coth\frac{\beta E}{2} - \frac{1}{2}\right\}, \tag{25}$$

where $x = z - z'$ and the arguments of $\epsilon(k)$ and $E(k)$ have been dropped for brevity. The colons denote normal ordering with respect to the field operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ (*not* with respect to the quasi-particle operators $b(k)$, $b^\dagger(k)$). By virtue of the identity

$$\int\frac{\mathrm{d}k}{2\pi}\frac{1 - \cos kx}{k^2} = \frac{|x|}{2} \tag{26}$$

an IR ($k \to 0$) divergence of the integrand translates into a 'diffusive spreading' of the quantum field $\Delta\psi$ as the distance x between two positions increases. This behavior appears only in the phase quadrature (imaginary part Y) which asymptotes to $\langle{:}Y^2{:}\rangle \sim D_Y|x - x'|$. The 'diffusion constant' is given by the simple and universal expression $D_Y = Mk_BT/\hbar^2 = \lambda^{-2}$.

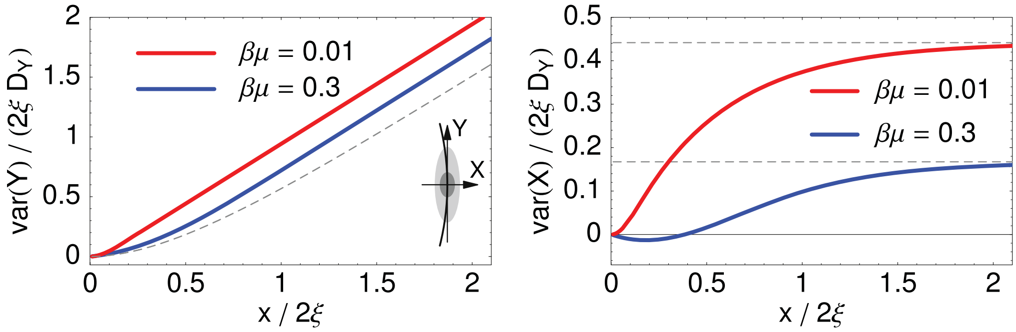

Figure 3. Spatial diffusion of quadrature components in Bogoliubov theory. We plot the variances of the imaginary (left) and real (right) parts of the field difference $\psi(z) - \psi(z') = X + \mathrm{i}Y$ versus the distance $x = z - z'$. Red (upper): low density. Blue (lower): high density, dashed gray: analytical approximation. The variances are calculated in normal order with respect to the ψ and $\psi^\dagger$ field operators and divided by the temperature-dependent 'diffusion constant' $D_Y = Mk_BT/\hbar^2$ to fit onto the same scale; the distance is scaled to the healing length ξ. Left: phase quadrature $\langle{:}Y^2{:}\rangle$. Blue: $\mu = 0.3\, k_BT$, red $\mu = 0.01\, k_BT$. The dashed line arises from the low-energy expansion (27). The inset illustrates the anisotropic diffusion of the complex field in the 'Mexican hat potential', starting from a symmetry-broken condensate value. Right: density quadrature $\langle{:}X^2{:}\rangle$. Blue: $\mu = 0.3\, k_BT$, red: $\mu = 0.01\, k_BT$. The dashed lines give the low-energy approximation (28) to the large-distance plateau. Negative values correspond to squeezing below the shot-noise level.

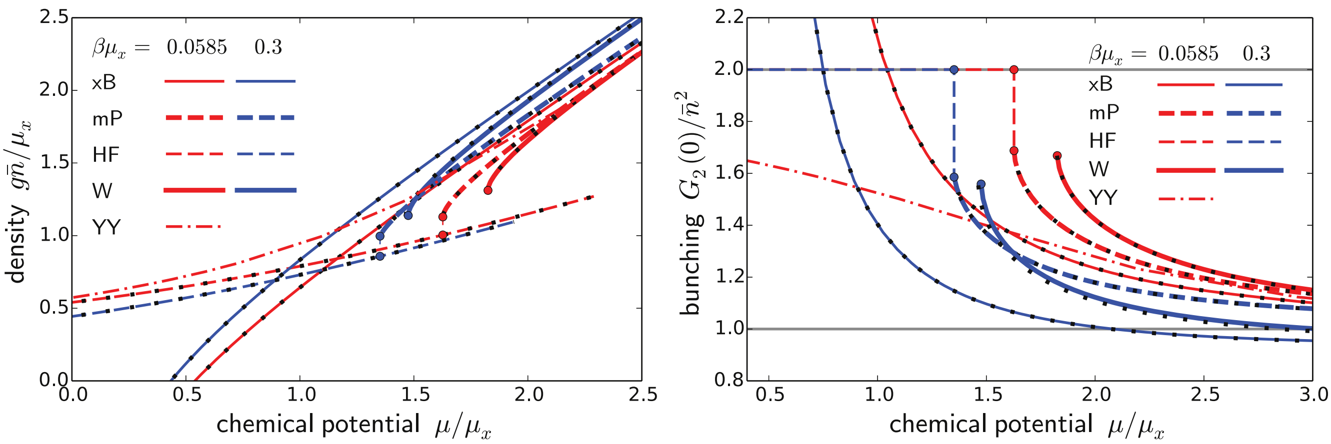

Figure 4. Equation of state (left) and density fluctuations (right). Comparison of different mean-field theories: thin solid lines (xB)—extended Bogoliubov theory (Mora and Castin, [25]); dashed lines (HF, mP)—modified Popov theory (Andersen, Al-Khawaja *et al*, [11, 12]); thick solid lines (W)—Hartree–Fock–Bogoliubov theory (Walser, [14, 16]). Dashed–dotted lines (YY)—finite-temperature solution to the Lieb–Liniger model (Yang and Yang, [39]). Cross-over units (see equation (2) for μ_x). Dotted (superimposed) lines: low-energy approximations. The black circles mark critical points predicted by certain mean-field theories. (left) Equation of state. Lower set of dashed curves: Hartree–Fock theory. (right) Density fluctuations, expressed via the pair correlation function $G_2(0)$. Upper line $G_2(0) = 2\bar{n}^2$: ideal gas and Hartree–Fock approximation. Lower line $G_2(0) = \bar{n}^2$: pure condensate. Anti-bunching ($G_2(0) < \bar{n}^2$) occurs in extended Bogoliubov theory for $\beta\mu \gtrsim 0.6$.

This is illustrated in figure 3(left) where the dashed line is calculated from the low-energy approximation

$$\int \frac{\mathrm{d}k}{2\pi}(1 - \cos kx)\frac{2\mu}{\beta E(k)^2} = D_Y\{|x| - \xi(1 - \mathrm{e}^{-|x|/\xi})\}. \tag{27}$$

with ξ the healing length (table 4).

The density quadrature (real part X) does not diffuse freely: its variance reaches a finite limit given by the integral in equation (25) with the cosine dropped. In appendix A.1, we find in the low-energy limit the result

$$|x| \gg \xi \gg \lambda: \qquad \langle{:}X^2{:}\rangle \approx \frac{\xi}{\lambda^2} - \frac{a_1}{\lambda} - \frac{a_2}{4}\frac{\lambda}{\xi^2}. \tag{28}$$

Note the close analogy of the sub-leading terms with the ideal gas expansion (8) where the same positive coefficients a_1, a_2 appear. As shown in figure 3(right), this agrees well with the (numerically computed) variance $\langle{:}X^2{:}\rangle$ at large distances. Note the negative values at low temperature ('below shot noise'): by analogy to quadrature fluctuations in quantum optics [60, 61], this can be interpreted as the squeezing of the density quadrature due to the nonlinear interaction with the condensate. At zero temperature, the squeezing reaches the level

$$T = 0,\ |x| \gg \xi: \quad \langle{:}X^2{:}\rangle \approx -\frac{1}{2\pi\xi}, \tag{29}$$

as an elementary integration shows (see equation (A10)).

3.1.3. Density correlations. We finally quote the density correlations $C(z - z')$ (see table 1). Due to the Bogoliubov shift, the density operator takes the form $\hat{n}(z) = |\phi|^2 + \phi^* \hat{\psi}(z) + \hat{\psi}^\dagger(z)\phi + \hat{\psi}^\dagger(z)\hat{\psi}(z)$, and we get additional contributions compared to the ideal gas (equation (9)). The second- and fourth-order correlations of the fluctuation $\hat{\psi}$ are worked out with the Wick theorem. The result can be written in the form

$$\begin{aligned} C(z - z') = {}& \bar{n}\,\delta(z - z') \\ & + 2\,\mathrm{Re}\,\{|\phi|^2 \langle \hat{\psi}^\dagger(z)\hat{\psi}(z')\rangle + \phi^{*2}\langle \hat{\psi}(z)\hat{\psi}(z')\rangle\} \\ & + |\langle \hat{\psi}^\dagger(z)\hat{\psi}(z')\rangle|^2 + |\langle \hat{\psi}(z)\hat{\psi}(z')\rangle|^2. \end{aligned} \tag{30}$$

This formula is only partially meaningful because the last two terms are both IR-divergent. The curly bracket can be combined into a regular integral

$$\begin{aligned} & 2\,\mathrm{Re}\,\{|\phi|^2 \langle \hat{\psi}^\dagger(z)\hat{\psi}(z')\rangle + \phi^{*2}\langle \hat{\psi}(z)\hat{\psi}(z')\rangle\} \\ & = 2|\phi|^2 \int \frac{\mathrm{d}k}{2\pi} \cos kx \left\{ \frac{\epsilon}{2E} \coth\frac{\beta E}{2} - \frac{1}{2} \right\}. \end{aligned} \tag{31}$$

This is essentially the same as the 'density quadrature' $\langle :X^2: \rangle$ (equation (25)). Figure 3(right) with a flip in orientation can thus be interpreted as a plot of the density correlation function. Note the density correlation length ξ that emerges from the typical k-scale $\epsilon(k) \sim \mu$ of the integrand.

The divergences of Bogoliubov theory have been addressed, of course, by the other mean-field theories we analyze now.

3.2. Extended Bogoliubov theory

Mora and Castin [25] have based this theory on an alternative expansion in the dense regime $\mu > 0$, using the assumption that phase gradients and density fluctuations are small. There is no spontaneous symmetry breaking here, but rather a (phase-fluctuating) quasi-condensate. For earlier work in this spirit in trapped systems, see [62]. The dilute side of the cross-over ($\mu < 0$) with significant bunching is excluded by construction, and one should expect that the expansion breaks down as the density is lowered.

3.2.1. Density and phase operators. The theory introduces a quasi-condensate component with density $n_q = \mu/g$ (denoted ρ_0 in [25]) and mutually conjugate phase and density fluctuation operators $\hat{\theta}(z)$, $\delta\hat{n}(z)$ [5, 62]. The existence of the phase operator is secured by working on a discrete lattice and assuming the probability of zero particles per lattice cell to be negligible. The Hamiltonian expanded to second order in the fluctuations can be diagonalized and yields again the Bogoliubov dispersion relation $E(k)$ (table 3) where μ/g appears in lieu of the condensate density n_c. With this proviso, the Bogoliubov amplitudes $u(k)$, $v(k)$ for the fluctuation operators have the same structure as in Bogoliubov theory (table 5). The mode expansions of the fluctuation operators are (k-arguments suppressed for simplicity)

$$\begin{aligned} \hat{\theta}(z) &= \hat{\varphi} + \frac{n_q^{-1/2}}{2\mathrm{i}} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} \{(u - v)b\,\mathrm{e}^{\mathrm{i}kz} - \mathrm{h.c.}\} \\ \delta\hat{n}(z) &= n_q^{1/2} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} \{(u + v)b\,\mathrm{e}^{\mathrm{i}kz} + \mathrm{h.c.}\}, \end{aligned} \tag{32}$$

where $\hat{\varphi}$ is the operator for the quasi-condensate phase (spatially constant). We have taken the thermodynamic limit where z is continuous and the momentum conjugate to $\hat{\varphi}$ can be neglected.

3.2.2. Equation of state. The average non-quasi-condensate density vanishes when computed with respect to the second-order Hamiltonian (subscript 2), $\langle \delta\hat{n} \rangle_2 = 0$. Third-order terms in the expansion are needed to describe the non-condensate density and are taken into account in perturbation theory. The resulting equation of state involves the same integrand as the density quadrature X in equation (25):

$$\mu = g\bar{n} + gn', \tag{33}$$

$$\begin{aligned} n' &= \int \frac{\mathrm{d}k}{2\pi} \{(u + v)^2 N(E) + v(u + v)\} \\ &= \int \frac{\mathrm{d}k}{2\pi} \left\{ \frac{\epsilon}{2E} \coth\frac{\beta E}{2} - \frac{1}{2} \right\}. \end{aligned} \tag{34}$$

This formula can be used to compute the mean density $\bar{n} = \bar{n}(\mu)$. Its structure is the same as in modified Popov theory (equation (46)), and we therefore used the notation n'. Mora–Castin theory does not pretend, however, that n' can be interpreted as a non-quasi-condensate density. Indeed, the integral (34) becomes negative at low temperatures. At zero temperature, we get (by the same calculation as in equation (29))

$$T = 0: \qquad \bar{n} = \frac{\mu}{g} + \frac{1}{2\pi\xi} \tag{35}$$

with the healing length $\xi = \hbar(4M\mu)^{-1/2}$. Since the first term is the quasi-condensate density n_q, the second one can be interpreted as the depletion density. In the opposite limit of high temperatures (low energies), we can use equation (28) to get

$$\bar{n} \approx \frac{\mu}{g} - \left\{ \frac{\xi}{\lambda^2} - \frac{a_1}{\lambda} - \frac{a_2}{4} \frac{\lambda}{\xi^2} \right\} \tag{36}$$

which is in excellent agreement with the data plotted in figure 4. One notes that the density exceeds the linear approximation $\bar{n} \approx \mu/g$ in the dense phase, this is due to the curly bracket in equation (36) becoming negative. We use cross-over units in this plot (see around equation (2)) and emphasize that despite the factor $\sim$130 in temperature, the scatter of the data is relatively small, also among the mean-field theories. In the dense phase, the convergence to Yang–Yang thermodynamics is quite excellent. Mora–Castin theory fails to predict a positive total density in the cross-over region, however: for $\mu \lesssim 2^{-2/3}\,\mu_x \approx 0.630\,\mu_x$ (see table 3). This could have been expected because in this range, density fluctuations become so large that the expansion around a 'quiet' quasi-condensate breaks down. The size of the density

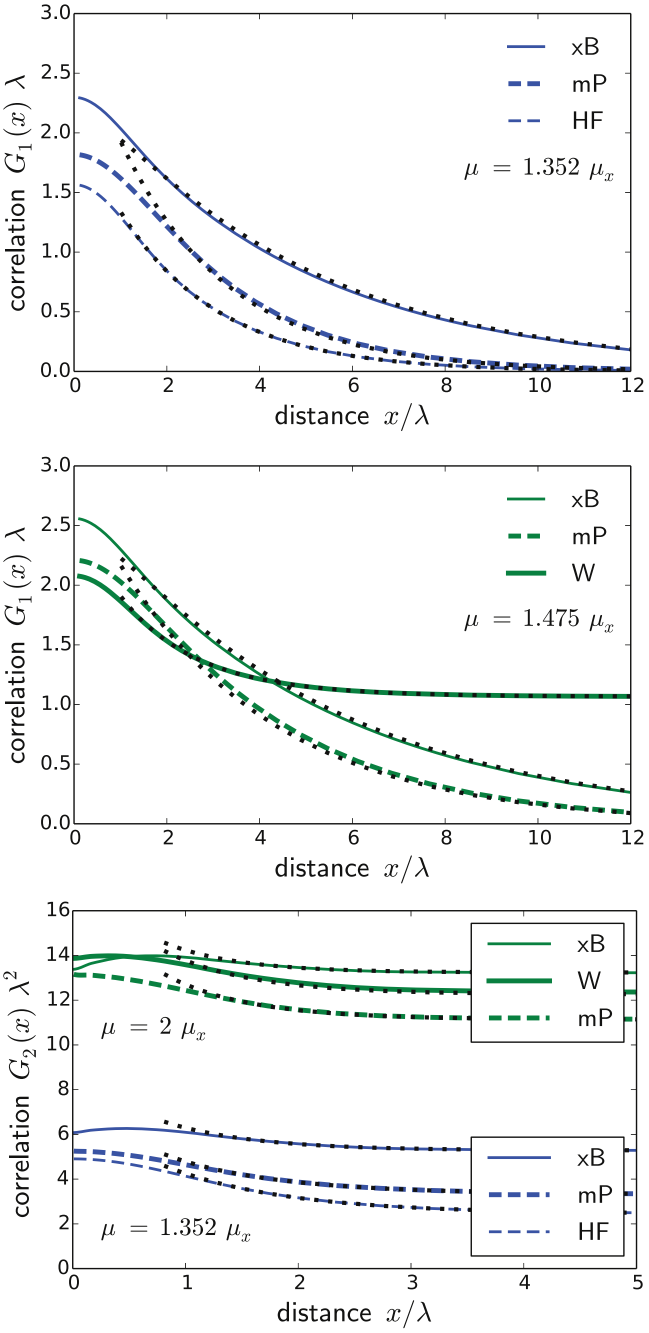

Figure 5. Comparison of correlation functions between mean field theories (labeling as in figure 4). (Top and center) Field correlation function $G_1(x)$ in the cross-over. Dotted curves: low-energy approximations. The distance is scaled to the thermal wavelength. Temperature such that $\beta\mu_x = 0.3$. (Top) Critical point of modified Popov theory where the description 'jumps' between 'HF' (dilute) and 'mP' (dense). (Center) Critical point of Hartree–Fock–Bogoliubov theory ('W' = Walser). Note the different behavior with respect to long-range order ('true versus quasi-condensate'). (Bottom) Pair correlation function $G_2(x)$ for two chemical potentials. Note the smaller range of distances x. Dotted lines: exponential approximations of equations (42), (53), (69), (71). Same temperature: $\beta\mu_x = 0.3$. Lower set of curves: critical point of modified Popov theory (mP/HF), below the range of Hartree–Fock–Bogoliubov theory (W).

fluctuations can be appreciated from the correlation functions in figures 4(right) and 5(bottom).

3.2.3. Correlation functions. The field correlation function is found as follows (equation (146) of [25])

$$G_1(x) = \bar{n}\exp\left[-\tfrac{1}{2}\langle{:}\Delta\hat{\theta}(x)^2{:}\rangle_2 - \frac{1}{8n_q^2}\langle{:}\Delta\delta\hat{n}(x)^2{:}\rangle_2\right], \tag{37}$$

where the difference operators $\Delta\hat{A}(x) = \hat{A}(x) - \hat{A}(0)$ are similar to the $\Delta\psi$ operator introduced around equation (25). The normal-order prescription $:\ldots:$ is with respect to the fluctuation operators $\hat{\psi}$. This expression includes in a perturbative way contributions to the Hamiltonian that are of third order in the fluctuations. The exponent in equation (37) has the convergent integral representation (equation (184) of [25])

$$\log\frac{G_1(x)}{\bar{n}} = -\frac{1}{\bar{n}}\int\frac{\mathrm{d}k}{2\pi}(1-\cos kx) \times\left\{\frac{\mu+\epsilon}{2E}\coth\frac{\beta E}{2} - \frac{1}{2}\right\}. \tag{38}$$

Mora and Castin [25] have recognized the integrand as the non-condensate spectrum of Bogoliubov theory (equation (22)). The IR divergence of the latter therefore yields an exponential decay at large distance x: $G_1(x) \sim \exp(-|x|/\ell_\theta)$ (using equation (27)). The (phase) correlation length $\ell_\theta = 2\bar{n}\lambda^2$ is twice as large as for the ideal Bose gas (parameter ℓ in equation (7)). A comparison to other mean-field theories is provided in figure 5(top, center).

At zero temperature, the integral (38) diverges only logarithmically. As explained in appendix A.3, one gets for large x (here, $\gamma \approx 0.577$)

$$T = 0: \qquad \log\frac{G_1(x)}{\bar{n}} \approx -\frac{\log(2|x|/\xi) + \gamma - 2}{4\pi\bar{n}\xi}. \tag{39}$$

The exponent of this power law has been given earlier by [5, 11], but even the prefactor agrees with [59] in the regime $\bar{n}\xi \gg 1$.

For later comparison with the modified Popov theory (section 4.1), we also quote the formula for phase diffusion. Keeping only terms up to second order, one gets indeed the phase quadrature $\langle{:}Y^2(x){:}\rangle$ of Bogoliubov theory(equation (25))

$$\langle{:}\Delta\hat{\theta}(x)^2{:}\rangle_2 = \frac{1}{n_q}\int\frac{\mathrm{d}k}{2\pi}(1-\cos kx)\left\{\frac{\epsilon+2\mu}{2E}\coth\frac{\beta E}{2} - \frac{1}{2}\right\}. \tag{40}$$

This term is at the origin of phase diffusion $\langle{:}\Delta\hat{\theta}(x)^2{:}\rangle_2 \approx |x|/(n_q\lambda^2)$ in the exponent of $G_1(x)$.

The density correlations are obtained directly from the expansion (32) of the fluctuation operator $\delta\hat{n}(z)$ (equation

(121) of [25])

$$C(z-z') = \bar{n}\delta(z-z') + \langle :\delta\hat{n}(z)\delta\hat{n}(z'):\rangle_2 \approx \bar{n}\delta(z-z') + 2\bar{n}n'(x). \tag{41}$$

Here, $n'(x)$ is given by equation (34) with an additional $\cos kx$ under the integral. Note that we recover the same expression as the regular part of equation (31) in Bogoliubov theory. The density correlation length is therefore of the order of the healing length ξ, much shorter than the characteristic phase correlation length ℓ_θ (see after equation (38)). A low-energy approximation to equation (41) can be found by keeping only the classical part $\coth(\beta E/2) \approx 2/(\beta E)$ of the integrand, leading to

$$x \gg \lambda: \qquad C(x) \approx 2\frac{\bar{n}\xi}{\lambda^2}\mathrm{e}^{-|x|/\xi}. \tag{42}$$

See figure 5(bottom) for a comparison. For $\mu \lesssim \mu_x$, density fluctuations are clearly too large for the expansion behind Mora–Castin theory to be valid. The squeezing of the density quadrature manifests itself by the non-monotonous behavior of the pair correlation function $G_2(x)$ as x increases from zero. At zero temperature, the density shows some 'anti-bunching'

$$T = 0: \qquad G_2(0) = \bar{n}^2 - \frac{\bar{n}}{\pi\xi} < \bar{n}^2, \tag{43}$$

but this small reduction is of course far from the 'correlation hole' of Fermi liquids or the Tonks–Girardeau gas [2, 63].

4. Self-consistent theories

These theories construct a simplified form for the Hamiltonian involving hydrodynamic fields. These are fixed at a later stage by equating them to thermodynamic averages computed with this approximate Hamiltonian ('self-consistency'). The simplest example of such a theory is Hartree–Fock (section 2.2) that works with a single field, the density $\bar{n}$. More elaborate methods also include a (quasi)condensate or, for example, the anomalous average, and aim at describing the Bose gas also at higher densities. We discuss here two examples in detail.

4.1. Modified Popov theory

This mean-field theory is based on the idea that low-energy fluctuations actually destroy the long-range order, and there is no condensate in the ordinary sense (long-range order à la Penrose–Onsager [1]). For details of the theory and similar approaches, we refer to [11–13, 62]. Note that in dimensions 2 and 3, the ('bare') interaction constant g gets renormalized into an energy- (and momentum-) dependent T-matrix [12, 64]. This effect is usually neglected in one-dimensional systems. We provide a brief discussion in section 4.1.3. The theory is applied differently on the two sides of the cross-over: on the dilute side, the Hartree–Fock approximation is applied (section 2.2), while the dense case is outlined now.

4.1.1. Equation of state. As the density increases beyond $\sim n_x$, the density $\bar{n} = n_q + n'$ of the system is split into the quasi-condensate n_q and the thermal part n'. The former determines the speed of sound in the (gapless) dispersion relation

$$E(k) = [2gn_q\epsilon(k) + \epsilon^2(k)]^{1/2}. \tag{44}$$

The thermal density is given by the convergent integral

$$n' = \int\frac{\mathrm{d}k}{2\pi}\left\{\frac{\epsilon}{2E}\coth\frac{\beta E}{2} - \frac{1}{2} + \frac{gn_q}{2(\epsilon+\mu)}\right\}, \tag{45}$$

where the first two terms have the same structure as equation (34). The last term has been introduced as a counterterm to regularize the zero-temperature (depletion) density. It has the merit of making equation (45) positive at all values of μ so that an interpretation as the density of the non-quasi-condensate is applicable. The equation of state is written

$$\mu = g\bar{n} + gn'. \tag{46}$$

It looks formally like equation (33) of extended Bogoliubov theory, although the interpretation of the non-condensate density n' is different. Equation (46) is an implicit equation for the chemical potential, since μ also appears in n'. See appendix B for details on the numerical procedure.

The zero-temperature analysis can be done similar to equation (35), and n' then describes the quasi-condensate depletion:

$$T = 0: \qquad n' \approx \frac{\pi/\sqrt{8} - 1}{2\pi\xi}, \tag{47}$$

where the approximation $\xi \approx \xi_q$ was used[4]. While the scaling with the healing length is the same, the prefactor differs from equation (35) due to the counterterm in equation (45). At high temperatures, the techniques of appendix A.2 can be used to derive the approximation

$$n' \approx \frac{\xi_q}{\lambda^2} - \frac{a_1}{\lambda} + \frac{\xi}{4\sqrt{2}\,\xi_q^2} - \frac{a_2\lambda}{4\xi_q^2}, \tag{48}$$

where $\xi_q = \hbar(4Mgn_q)^{-1/2}$ is the healing length of the quasi-condensate density n_q.

In figure 4(left), the equation of state (thick dashed) is compared to extended Bogoliubov theory (thin solid). In the dense phase, the difference is small, the self-consistent theory predicts a slightly lower density. In the cross-over region $\mu \approx 1.89\,\mu_x$, a 'critical point' is reached (see table 3): below this value, the implicit equation of state has no solution. There is a finite gap to the density given by Hartree–Fock theory (lower lines), which is the appropriate mean-field description on the dilute side [12]. In this regime, a smooth connection to Yang–Yang thermodynamics is found.

4.1.2. Correlation functions. The first-order correlation function can be found, e.g., in equation (8) of [13]

$$G_1(x) = \bar{n}\,\exp\left[-\tfrac{1}{2}\langle\Delta\theta^2(x)\rangle_{\mathrm{mP}}\right], \tag{49}$$

[4] There is probably a misprint in [11] and in equation (26) of [12] where the denominator is $4\pi\xi$.

it involves phase fluctuations given by (subscript for 'modified Popov')

$$\langle\Delta\theta^2(x)\rangle_{\mathrm{mP}} = \frac{1}{n_q}\int\frac{\mathrm{d}k}{2\pi}(1-\cos kx)\left\{\frac{gn_q}{E}\coth\frac{\beta E}{2} - \frac{gn_q}{\epsilon+\mu}\right\}. \quad (50)$$

The first term in curly brackets is proportional to the anomalous average of Bogoliubov theory (24), the second one is the same counterterm as in the thermal density n' (equation (45)) and makes the integral converge in the UV. The IR singularity of the integrand is the same as in Mora–Castin theory (40), so that at large distances, a similar phase diffusion is found: $\langle\Delta\theta^2(x)\rangle \approx |x|/\ell_\theta$ with $\ell_\theta = n_q\lambda^2$. The phase coherence length hence grows linearly with the quasi-condensate density. The plots in figure 5(top, center) illustrate that the difference $n_q < \bar{n}$ makes the predicted phase coherence somewhat smaller than in extended Bogoliubov theory (thin solid). Hartree–Fock theory is even less coherent, as shown in the top panel.

At zero temperature, the phase fluctuations are sub-diffusive and increase logarithmically ($|x| \gg \xi \approx \xi_q$)

$$T=0: \quad \langle\Delta\theta^2(x)\rangle_{\mathrm{mP}} \approx -\frac{\log(2|x|/\xi)+\gamma-\pi/\sqrt{2}}{2\pi\bar{n}\xi}. \quad (51)$$

The power law that this implies for $G_1(x)$ (equation (49)) has the same exponent as equation (39), but a slightly different prefactor.

Finally, to come to density correlations, we note that equation (30) is also valid in the presence of a quasi-condensate as long as one assumes that the fluctuations obey Gaussian statistics. We generalize slightly the expressions of [11–13] to cover the case $z \neq z'$: as explained around equation (37) in [12], the anomalous averages are removed from equation (30), and one gets (figure 5(bottom))

$$G_2(x) = \bar{n}^2 + 2n_q n'(x) + [n'(x)]^2. \quad (52)$$

Here, the function $n'(x)$ is given by equation (45) with an additional factor $\cos kx$ inserted under the integral. As noted in [13], the reduction of density fluctuations, relative to the ideal gas, provides an alternative interpretation of the quasi-condensate density: $G_2(0) = 2\bar{n}^2 - n_q^2$. On the other hand, since $n_q \leqslant \bar{n}$ by construction, one always has $G_2(0) \geqslant \bar{n}^2$, and there is no possibility for anti-bunching in modified Popov theory (see figure 4(right)).

The density correlations can be approximated quite accurately (dotted lines in figure 5(bottom)) by using

$$x \gg \lambda: \qquad n'(x) \approx \frac{\xi_q\,\mathrm{e}^{-|x|/\xi_q}}{\lambda^2} + \frac{\xi\,\mathrm{e}^{-|x|/(\sqrt{2}\,\xi)}}{\sqrt{32}\,\xi_q^2}. \quad (53)$$

The first term results in a formula similar to equation (42), but involving the quasi-condensate healing length ξ_q. The second is small at low energies and arises from the counter term.

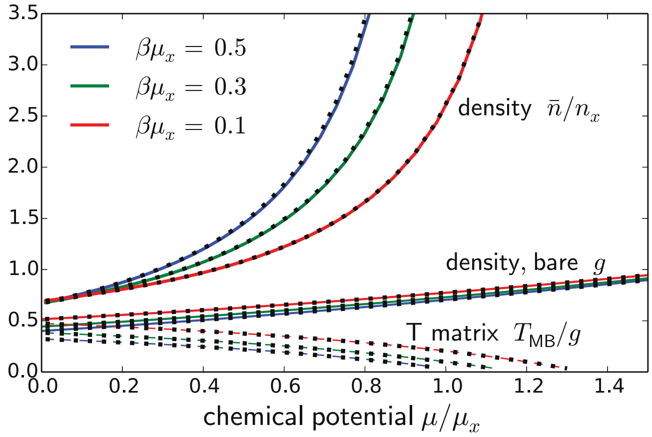

Figure 6. Illustration of renormalized interactions due to many-body effects for Hartree–Fock theory. Upper set of lines: density based on self-consistent T-matrix (56). Center set: comparison to density in 'bare' Hartree–Fock theory (i.e., constant coupling g). Lower curves: renormalized T-matrix. Dotted curves: low-energy approximations.

4.1.3. Renormalised interactions. The scattering between two atoms in a dense gas occurs in a 'background field' formed by the other atoms. This leads to an energy- and density-dependent change in the matrix elements of the interaction potential [65]. For completeness, we discuss here the formulas given in [12], adapted to our notation.

As a first example, consider two atoms in the condensate that collide at zero temperature. The bare interaction constant g is replaced by the two-body T-matrix element (equation (7) of [12])

$$\frac{1}{T_{2\mathrm{B}}(-2\mu)} = \frac{1}{g} + \int\frac{\mathrm{d}k}{2\pi}\frac{1}{2(\epsilon+\mu)}, \quad (54)$$

where the denominator involves the pair's kinetic energy and the change in the condensate energy as two atoms are removed. This integral evaluates to (see equation (A12))

$$T_{2\mathrm{B}}(-2\mu) = \frac{g}{1+g/(\sqrt{32}\,\mu\xi)} \quad (55)$$

and illustrates that the interactions renormalize to zero as $\mu \to 0$. The magnitude of this effect is small in practical one-dimensional systems because the denominator involves the small Lieb–Liniger parameter $g/(2\mu\xi) \sim (n_c\xi)^{-1/2} \sim \gamma^{1/4}$.

Our second example are the thermal corrections to the scattering matrix. Consider for simplicity the dilute phase and the many-body effects in Hartree–Fock theory. The average density is worked out as in equation (11), with the mean-field shift of the chemical potential replaced by the Hartree–Fock self-energy, $2g\bar{n} \mapsto \Sigma$. According to equation (29) of [12], the renormalised T-matrix is

$$\frac{1}{T_{\mathrm{MB}}(-\Sigma)} = \frac{1}{g} + \int\frac{\mathrm{d}k}{2\pi}\frac{\coth\frac{1}{2}\beta[\epsilon+\Sigma-\mu]}{2\epsilon+\Sigma}, \quad (56)$$

where equation (54) has been used. The equations are closed by the self-consistency relation $\Sigma = 2\bar{n}T_{\mathrm{MB}}(-\Sigma)$, equation (28) of [12].

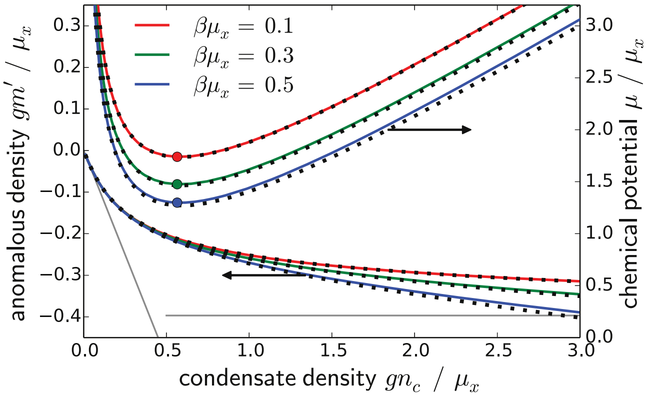

Figure 7. Anomalous average and chemical potential in Walser's mean field theory (cross-over units). Lower set of lines (left scale): anomalous average m', plotted versus the condensate density. The gray lines correspond to $m' = -n_c$ and $gm' = -2^{-4/3}\,\mu_x \approx -0.397\,\mu_x$. Solid: iterative solution based on numerical integration of equation (65); dotted: low-energy approximation (67). Upper curves (right scale): chemical potential (60) based on the self-consistent value for m'. The solid lines are integrated numerically, based on equation (64); the dotted lines are computed from the approximation (66). The colored dots mark the 'critical point' below which no solution is found.

A numerical solution is shown in figure 6 and illustrates that a critical point appears at $\mu \sim \mu_x$ (the precise value depends on $\beta\mu_x$), where the many-body interactions renormalize to zero and the density diverges. The low-energy approximation for equation (56) is (from the techniques of appendix A.2)

$$\frac{1}{T_{\mathrm{MB}}(-\Sigma)} \approx \frac{1}{g} + \frac{k_B T}{\sqrt{(\Sigma/\mu - 1)\Sigma}} \frac{1/(2\mu\xi)}{\sqrt{\Sigma/2} + \sqrt{\Sigma - \mu}} + \frac{2a_2}{k_B T \lambda}. \tag{57}$$

This gives, in conjunction with equation (8) for the density, a relatively accurate picture (dotted lines in figure 6). Note the strong (≈50%) reduction of interactions already for $\mu = 0$. We find in particular that the self-energy approaches $\Sigma \to \mu$ at the critical point.

4.2. Hartree–Fock–Bogoliubov

The Hamiltonian is approximated in this mean-field theory by the quadratic expression

$$H \approx \epsilon_c L + \int \mathrm{d}z \left\{ \frac{\hbar^2}{2M} \frac{\mathrm{d}\hat{\psi}^\dagger}{\mathrm{d}z} \frac{\mathrm{d}\hat{\psi}}{\mathrm{d}z} + (2g\bar{n} - \mu)\hat{\psi}^\dagger\hat{\psi} + \frac{g}{2}(m^*\hat{\psi}^2 + m(\hat{\psi}^\dagger)^2) \right\}, \tag{58}$$

where $\hat{\psi}$ is again the fluctuation operator around a condensate field ϕ. The first term is the condensate energy, the first piece under the integral formally identical to Hartree–Fock theory (equation (10)), the total density being split into $\bar{n} = |\phi|^2 + n'$. The last terms involve the anomalous average $m = \phi^2 + m'$ that already appeared in Bogoliubov theory (equation (24)). This Hamiltonian is complemented by the generalized Gross–Pitaevskii equation for the condensate field ϕ

$$-\frac{\hbar^2}{2M}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + g(\bar{n} + n')\phi + gm'\phi^* = \mu\phi. \tag{59}$$

A derivation of these equations has been discussed by Griffin [9] who also uses the name 'Hartree–Fock–Bogoliubov theory'. Keeping the anomalous average in full goes back to Girardeau and Arnowitt (see [17] for a discussion in three dimensions).

In a homogeneous system with real ϕ, one finds the equation of state

$$\mu = g(\bar{n} + n') + gm' = g|\phi|^2 + g(2n' + m'). \tag{60}$$

The anomalous average $m' < 0$ reduces the chemical potential relative to extended Bogoliubov and to modified Popov theory (equations (33), (46)). This has also been interpreted as a many-body-induced reduction of the particle interactions [7, 65].

The expansion of the operator $\hat{\psi}$ is the same as in Bogoliubov theory (15), but the amplitudes $u = u(k)$, $v = v(k)$ solve the modified system

$$\begin{pmatrix} \epsilon + 2g\bar{n} - \mu & gm \\ gm^* & \epsilon + 2g\bar{n} - \mu \end{pmatrix}\begin{pmatrix} u \\ v^* \end{pmatrix} = \begin{pmatrix} Eu \\ -Ev^* \end{pmatrix}. \tag{61}$$

One gets the dispersion relation (using equation (60)):

$$E = ((\epsilon + 2g\bar{n} - \mu)^2 - g^2|m|^2)^{1/2} \tag{62}$$

$$= \sqrt{(\epsilon - 2gm')(\epsilon + 2g|\phi|^2)}\,. \tag{63}$$

The dispersion relation has the particular feature that it shows a finite gap, $E(k \to 0) = 2g|\phi|\sqrt{-m'}$. Walser argues, in particular in [16], that the gap is not in contradiction with the existence of a Goldstone mode due to the U(1)-symmetry of the original theory. The Bogoliubov modes found here are a 'convenient quasi-particle basis' to describe the thermodynamic equilibrium state of the Bose gas. The finite gap is essential here to regularize the theory in the IR. For the linear response of a perturbation to the gas, a different calculation is performed that leads, indeed, to a gapless spectrum of collective excitations. The key difference is that the perturbation also affects the condensate phase which is treated as a dynamical variable, rather than fixed to a symmetry-broken value [16].

We note that in the self-consistent HFB theory of Yukalov and Yukalova [17, 19], a gapless dispersion relation for quasi-particles is constructed by introducing a second chemical potential (for the non-condensate particles). Most of this analysis focuses on three dimensions, however. We do not discuss this variant further here because when formulas are extrapolated to the one-dimensional setting, one finds IR-divergent expressions similar to Bogoliubov theory (section 3.1). The dimensional regularization suggested in [19] leads to a negative non-condensate density, similar to extended Bogoliubov theory (section 3.2).

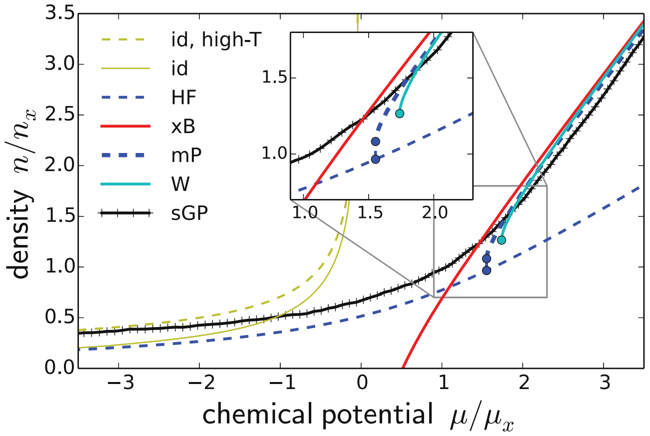

Figure 8. Comparison of equation of state between mean-field theories (cross-over units). The temperature is fixed so that $\beta\mu_x = 0.1$. The inset provides a zoom into the 'critical region'. Labels: id, high-*T*: ideal gas, first term in the low-energy expansion of equation (8); id: ideal gas; HF: Hartree–Fock; xB: extended Bogoliubov theory of Mora and Castin [25]; mP: modified Popov theory of Andersen, Al Khawaja *et al* [11, 12], W: Hartree–Fock–Bogoliubov theory, as developed by Walser [9, 16]; sGP: classical field simulation of the stochastic Gross–Pitaevskii equation [2, 27, 28, 30, 33], using the parameters of table 2, left column.

4.2.1. Mean-field densities. The parameters n' and m' are determined by consistency from the moments of the fluctuation operator $\hat{\psi}$ in the gaussian ensemble defined by equation (58), for example $\langle\hat{\psi}^\dagger\hat{\psi}\rangle = n'$. This yields again equation (21) as in Bogoliubov theory, but since the expressions for the amplitudes u, v are different, the resulting integral is regular

$$n' = \int\frac{\mathrm{d}k}{2\pi}\left\{\frac{\epsilon + g(n_c - m')}{2E}\coth\frac{\beta E}{2} - \frac{1}{2}\right\}. \tag{64}$$

Similarly, for the anomalous average $\langle\hat{\psi}\hat{\psi}\rangle = m'$, one finds

$$m' = -\int\frac{\mathrm{d}k}{2\pi}\frac{g(n_c + m')}{2E}\coth\frac{\beta E}{2}. \tag{65}$$

This is an implicit equation since m' also appears in the mode energies E (equation (63)). The $T = 0$ limit has been evaluated in [66] in terms of elliptic integrals[5]. It has been shown that the behavior of the condensate depletion is qualitatively similar to equations (35), (47), except for a logarithmic correction $\sim\log(\bar{n}\xi)/\xi$ (equation (44) of [66]).

In the opposite limit of high temperatures (low energies), the techniques sketched in appendix A.2 yield (correcting one sign in table 7.1 of [57])

$$n' \approx \frac{\xi_c}{2\lambda^2}\left(\sqrt{\frac{n_c}{-m'}} + 1\right) - \frac{a_1}{\lambda} + \frac{a_2\lambda}{4\xi_c^2}\left(1 + \frac{m'}{n_c}\right), \tag{66}$$

$$m' \approx -\frac{\xi_c}{2\lambda^2}\left(\sqrt{\frac{n_c}{-m'}} - 1\right) - \frac{a_2\lambda}{2\xi_c^2}\left(1 + \frac{m'}{n_c}\right), \tag{67}$$

where ξ_c is the healing length corresponding to the condensate mean field gn_c and a_1, a_2 the zeta-function coefficients introduced earlier (equation (8)). In figure 7, this approximation is used to find the anomalous average, yielding the dotted lines. A coarse estimate for large n_c (horizontal line) can be found by keeping only the first term in equation (67). For small n_c, we note $m' \approx -n_c$.

The chemical potential μ is plotted in the same figure as a function of the condensate density n_c. One notes that there is a 'critical' chemical potential $\mu_{\mathrm{cr}} = \mu_{\mathrm{cr}}(\beta)$ below which the equations have no solution (colored dots). To see this intuitively, recall that small values of n_c and m' make for a large non-condensate density n' because of the near-divergence in the IR (the first term $\sim n_c^{-1/2}$ in equation (66)). As n_c grows, it eventually dominates in the chemical potential $\mu = g(n_c + 2n' + m')$, so that the latter must go through a minimum, which is located in the range $\mu \sim \mu_x$. The same phenomenon occurs in modified Popov theory (figure 4), only the exact location of the 'critical point' is different.

4.2.2. Correlation functions. The correlation function of the field operator is

$$G_1(x) = n_c + n'(x), \tag{68}$$

where $n'(x)$ is given by equation (64) with an additional factor $\cos kx$ under the integral. Due to the gapped dispersion relation, this is regular in the IR. It shows long-range order at the level of the condensate, $G_1(x \to \infty) = n_c$ (figure 5(center)). Quite different from the previous theories, this version of Hartree–Fock–Bogoliubov theory thus predicts a true condensate (even at one-dimension). The non-condensate contribution has, in the leading order, the low-energy (and large-distance) approximation (dotted curve in the figure)

$$n'(x) \approx \frac{\xi_c}{2\lambda^2}\left(\sqrt{\frac{n_c}{-m'}}\,\mathrm{e}^{-|x|/\xi_m} + \mathrm{e}^{-|x|/\xi_c}\right), \tag{69}$$

where $\xi_m \sim (-m')^{-1/2}$ may be called the 'anomalous healing length' (see table 4).

For density fluctuations in Walser's mean field theory, we may use equation (30) because it is based on a Gaussian equilibrium ensemble. In distinction to conventional Bogoliubov theory, all integrals are convergent, and we get for the pair correlation function

$$G_2(x) = \bar{n}^2 + 2n_c[n'(x) + m'(x)] + [n'(x)]^2 + [m'(x)]^2 \tag{70}$$

(for $m'(x)$, insert $\cos kx$ under the integral (65)). The analog of the large-distance approximation (69) is

$$m'(x) \approx -\frac{\xi_c}{2\lambda^2}\left(\sqrt{\frac{n_c}{-m'}}\,\mathrm{e}^{-|x|/\xi_m} - \mathrm{e}^{-|x|/\xi_c}\right). \tag{71}$$

When this is inserted into the density correlation function (70), it compares quite well with the numerical calculations, see figure 5(bottom).

[5] Equation (52) of [66] corrects a typographic error in equation (91) of [16].

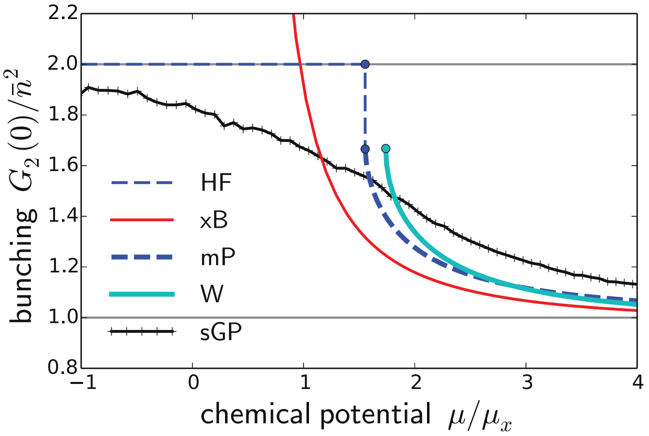

Figure 9. Particle bunching (pair correlations $G_2(0)$), normalized to the average density squared. Comparison between mean-field theories and classical field simulations (black line with markers). Temperature such that $\beta\mu_x = 0.1$. Labeling as in figure 8.

5. Discussion

5.1. Complex field simulations and parameters

We use the physical parameters collected in table 2, left column. They correspond to a dimensionless inverse temperature $\beta\mu_x \approx 0.1$. In the following plots, we add for comparison the results of classical field simulations (sGP equation [2]) which were performed for a trapped system with these parameters and the shallow trap frequency $1.4\,\mathrm{Hz} \approx 0.1\,\mu_x/h$. A real-space grid with spacing $\Delta z \approx 0.5\,\lambda$ was used, slightly larger than the value proposed in extended Bogoliubov theory, $\Delta z = \xi\beta\mu$ (equation (176) of [25]). If we estimate with Hartree–Fock theory the density of atoms at energies above the cutoff $E_{\max} \sim \hbar^2/(M\Delta z^2)$ (not captured by the simulations), we find a negligible contribution for $\mu \sim 0$. We plot the data against the local chemical potential $\mu - V(z)$, assuming the LDA is valid. A discussion of this assumption is provided in appendix C.

5.2. Equation of state

In figure 8, we compare the equations of state for all mean-field theories discussed so far. Coming from negative μ, Hartree–Fock allows to enter smoothly the cross-over region. It fails by 50% in the dense phase, however. One might switch to extended Bogoliubov theory (xB) at $\mu \sim \mu_x$ where the two equations of state cross, but this prescription is lacking a more detailed justification. The self-consistent modified Popov theory involves a finite jump in the density when one switches from HF to its 'end point' (inset of figure 8). A similar jump appears when the HFB theory proposed by Walser is taken. In the dense phase, the three mean-field theories converge fairly well, but the xB density is systematically higher (see also figure 4(left)). Note that the stochastic simulation (sGP) is able to describe the density smoothly throughout the cross-over. Its only deficiency appears in the dilute phase where it joins the classical (low-energy or Rayleigh–Jeans) approximation instead of the full (Bose–Einstein) prediction of the ideal gas. This had to be expected because the noise term (Langevin force) in the sGP is used in the classical (high-temperature) approximation and becomes dominant in the dilute regime.

5.3. Density fluctuations

A survey of the predictions for density fluctuations is given in figure 9. We plot the normalized pair correlation function $g_2 = G_2(0)/\bar{n}^2$. The ideal gas and Hartree–Fock theory give a value of 2 typical for a complex Gaussian (or chaotic) field (Wick theorem). In modified Popov theory, this jumps at a 'critical chemical potential' $\mu \sim 1.5\,\mu_x$ down to a value $g_2 \sim 1.6$, and decreases for $\mu > 1.5\,\mu_x$ steeply to the 'pure condensate' value $g_2 = 1$. From its critical point on (which is slightly shifted), Walser's HFB theory behaves similarly. In the extended Bogoliubov theory, the density fluctuations diverge as $\mu \to 0$, and one clearly leaves its region of validity. The stochastic simulation behaves again smoothly and shows that the suppression of density fluctuations is already significant on the dilute side of the cross-over ($\mu < 0$).

5.4. Conclusion

We have analyzed the cross-over of a weakly interacting, homogeneous Bose gas in one dimension in the thermodynamic limit. Interactions (repulsive) stabilize the dilute phase, as the chemical potential increases above zero, but at finite temperature, phase fluctuations persist in the dense phase and preclude any long-range order (quasi-condensate). Using a suitable thermodynamic scaling of the relevant variables, the cross-over can be mapped to a relatively narrow range of reduced variables, e.g., $-\mu_x \lesssim \mu \lesssim 3\,\mu_x$ where $\mu_x \sim (gT)^{2/3}$. We have worked through a portfolio of mean field theories to describe the cross-over. Hartree–Fock theory performs better than the ideal gas model, but fails to capture the equation of state and the reduction of density fluctuations in the quasi-condensed phase. This does not seem to improve when the many-body renormalization of atomic interactions is taken into account. The extended Bogoliubov theory of Mora and Castin breaks down when the cross-over is approached from the dense side, because density fluctuations become too strong. Self-consistent theories (modified Popov of Stoof *et al*, Hartree–Fock–Bogoliubov of Walser, Holland *et al*) predict a critical point in the equation of state because IR divergences at low (quasi)condensate density enforce a minimal value for the chemical potential. The failure appears for both gapped and gapless quasi-particle spectra. The issue of constructing a number-conserving theory (fixed particle number, canonical ensemble) is of minor importance for the homogeneous system we were focusing on. It can be checked explicitly from [25] that the specific features (projection of quasi-particle modes perpendicular to the condensate, condensate phase operator) become irrelevant in the thermodynamic limit.

We could gauge this state of affairs by comparison to two successful models for the cross-over. One is provided by the exact solution of the (Lieb–Liniger) Yang–Yang equations, which gives an easy access to low moments of the density [40, 49, 52]. The second method builds on complex-field simulations (stochastic Gross-Pitaevskii equation) that capture the low-lying modes of the quasi-condensate which can be described classically. With a suitable choice of numerical cutoff, these simulations are essentially unique. Their smooth density profiles through the cross-over region have already compared favorably with experiments. We may expect that the distribution functions (counting statistics) that can be extracted from them (see, e.g., [67]) may help curing the deficiencies of mean field theories. We have reasons to believe that the failures of mean-field are related to the break-down of the Gaussian approximation to the probability distribution of the quantum field. (For a discussion of beyond-Gaussian correlations in c-field methods, see [68].) This conclusion is based on the comparison with a classical field theory which will be reported elsewhere [69].

Acknowledgments

T-OS thanks the University of Newcastle and the Joint Quantum Centre Durham-Newcastle for their hospitality, and Vanik E Mkrtchian and Timo Felbinger for various help at University of Potsdam. The stochastic simulations reported here were performed on the Condor cluster at Newcastle. We thank K Kheruntsyan for sharing results of numerical calculations of the Yang–Yang model. We thank Antonio Negretti and Hansjörg Polster (C.H.) and Stuart Cockburn (N.P) for helpful discussions in various stages of this work. Gratefully acknowledged is financial support from the *Deutsche Forschungsgemeinschaft* (grant nos. Schm-1049/7-1 and Fo 703/2-1, C.H.) and the EPSRC (grant no. EP/F055935/1, N. P.). Data supporting this publication is openly available under an 'Open Data Commons Open Database License'. Additional metadata are available at: http://doi.org/10.17634/072805-1. Please contact Newcastle Research Data Service at rdm@ncl.ac.uk for access instructions.

Appendix A. Low-energy expansions

A.1. Bose function

The Bose function is also known as polylogarithm $\mathrm{Li}_\nu(\mathrm{e}^x)$:

$$\mathrm{g}_\nu(x) = \sum_{n=1}^{\infty} \frac{\mathrm{e}^{nx}}{n^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}t \frac{t^{\nu-1}}{\mathrm{e}^{t-x} - 1}. \tag{A1}$$

The sum converges only for $x < 0$ or a fugacity $\mathrm{e}^x < 1$. Of interest here is the case $\nu = 1/2$ and the 'high-temperature expansion' approaching the critical point from below[6]

$$x \nearrow 0: \qquad \mathrm{g}_{1/2}(x) \approx \sqrt{\frac{\pi}{-x}} + \zeta\left(\tfrac{1}{2}\right) + \zeta\left(-\tfrac{1}{2}\right)x + \mathcal{O}(x^2) \tag{A2}$$

[6] Equation (25.12.12) in Digital Library of Mathematical Functions, `dlmf.nist.gov`.

with coefficients given by the (analytically continued) Zeta function. The first term can be found by expanding the exponential under the integral (A1). Subtracting this convergent integral and expanding the integrand for small x, we observe that the lowest terms provide convergent integrals. They yield the following integral representations for the ζ-coefficients

$$\int_0^\infty \frac{\mathrm{d}q}{\pi} \left\{ \frac{1}{\mathrm{e}^{q^2/2} - 1} - \frac{2}{q^2} \right\} = \frac{\zeta\left(\frac{1}{2}\right)}{\sqrt{2\pi}} = -a_1, \tag{A3}$$

$$\int_0^\infty \frac{\mathrm{d}q}{\pi} \left\{ \frac{1}{4\sinh^2(q^2/4)} - \frac{4}{q^4} \right\} = \frac{\zeta\left(-\frac{1}{2}\right)}{\sqrt{2\pi}} = -a_2, \tag{A4}$$

where the suggestive substitution $t = q^2/2$ was made. The coefficients are approximately $a_1 \approx 0.5826$, $a_2 \approx 0.0830$.

A.2. High temperature expansion

As an illustration of the technique, we consider the integral that appears in the non-condensate density (45) (see also equation (25))

$$I_1(\beta) = \int \frac{\mathrm{d}k}{2\pi} \left\{ \frac{(k^2/2)\coth\left(\frac{1}{2}\beta E(k)\right)}{2E(k)} - \frac{1}{2} \right\} \tag{A5}$$

$$\approx \frac{1}{2\beta} - \frac{a_1}{\sqrt{\beta}} - a_2\sqrt{\beta} \qquad (\beta \to 0). \tag{A6}$$

To simplify the notation in this appendix, we use units where the Bogoliubov dispersion relation is $E(k) = |k|\sqrt{1 + k^2/4}$. The dimensionless inverse temperature is $\beta = Mc^2/k_B T$ with the speed of sound c.

The integrand is even k, and we restrict to $0 \leqslant k < \infty$. Convergence in the IR is secured by the 'coherence factor' $k^2/(4E(k))$ in front of the hyperbolic cotangent. The classical (high-temperature) limit of the latter integrates to the first term in equation (A6)

$$\int_0^\infty \frac{\mathrm{d}k}{\pi} \frac{k^2/2}{\beta E^2(k)} = \frac{1}{2\beta}. \tag{A7}$$

The next order arises when this classical limit is subtracted from the integrand, and the high-energy approximation $E(k) \approx k^2/2$ is applied:

$$\int_0^\infty \frac{\mathrm{d}k}{\pi} \left\{ \frac{1}{\mathrm{e}^{\beta k^2/2} - 1} - \frac{1}{\beta k^2/2} \right\} = -\frac{a_1}{\sqrt{\beta}} \tag{A8}$$

using the substitution $q = \sqrt{\beta}\, k$ and the identity (A3). Note that this also includes the last term $-1/2$ ('vacuum subtraction') from equation (A5).

When the terms in equations (A7), (A8) are subtracted from the integrand, we get an expression that is still integrable both at low and high momentum. We perform again the substitution $q = \sqrt{\beta}\, k$ and expand (at fixed q) for small β. The dispersion relation, for example, becomes $\beta E(k) = \frac{1}{2}q(q^2 + 4\beta)^{1/2} \approx \frac{1}{2}q^2 + \beta + \mathcal{O}(\beta^2/q^2)$. The resulting integral scales like $\beta^{1/2}$ and, in the leading order, involves the integrand

$$
\begin{aligned}
&-\sqrt{\beta}\,\frac{(q^2-8)\mathrm{e}^{q^2/2}+(q^4+16)-(q^2+8)\mathrm{e}^{-q^2/2}}{4q^4\sinh^2(q^2/4)}\\
&=-\sqrt{\beta}\left\{\frac{1}{4\sinh^2(q^2/4)}-\frac{8}{q^4}+\frac{\coth(q^2/4)}{q^2}\right\}.
\end{aligned}
\tag{A9}
$$

The second form makes the subtractions quite transparent that regularize the integrand as $q\to 0$. The first and one half of the second term yield $a_2\sqrt{\beta}$ from equation (A4). The remainder is integrated by parts to make the derivative of the coth appear, taking care of the cancelling poles. We again find the integral of equation (A4), but with a different prefactor: $-2a_2\sqrt{\beta}$. The sum gives the last term in equation (A6).

The next order in this expansion would be $\mathcal{O}(\beta^{3/2})$. In modified Popov theory, the last term in the non-condensate density (45) is integrated elementarily. Since it is temperature-independent, it 'slips' between the a_1 and a_2 terms in equation (A6).

A.3. Zero-temperature expansion

The non-condensate density involves two integrals. The first one is

$$
I_a=\int\frac{\mathrm{d}k}{2\pi}\frac{\epsilon-E}{2E}. \tag{A10}
$$

By adopting the units explained after equation (A5)), a dimensional factor $1/(2\xi)$ is pulled out. Here, $\xi=\hbar\,(4M\mu)^{-1/2}$ for extended Bogoliubov theory and $\xi\mapsto\xi_q$ for modified Popov. Make the substitution $k=2\sinh t$ and get

$$
\begin{aligned}
&\epsilon=2\sinh^2 t,\qquad E=2\sinh|t|\cosh t\\
&I_a=\frac{1}{2\pi\xi}\int_0^\infty \mathrm{d}t\,(\sinh t-\cosh t)=-\frac{1}{2\pi\xi}.
\end{aligned}
\tag{A11}
$$

The second piece is the term:

$$
I_b=\int\frac{\mathrm{d}k}{2\pi}\frac{gn_q}{2(\epsilon+\mu)} \tag{A12}
$$

which reduces in our units with $k=\sqrt{2}\,q$ to

$$
I_b=\frac{gn_q}{2\pi\sqrt{2}\,\mu\xi}\int_0^\infty\frac{\mathrm{d}q}{q^2+1}=\frac{1}{2\pi\xi_q}\frac{\pi\xi}{\xi_q\sqrt{8}}. \tag{A13}
$$

The zero-point energy density of Bogoliubov theory, equation (17), is integrated similarly. After the substitution $k=\xi^{-1}\sinh t$,

$$
\epsilon_0-\epsilon_c=-\frac{\mu}{2\pi\xi}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-2t}\cosh t=-\frac{\mu}{3\pi\xi} \tag{A14}
$$

which is equation (18).

Some correlation functions involve the integral

$$
C_1(x)=\int\frac{\mathrm{d}k}{2\pi}(1-\cos kx)\frac{\mu}{E(k)}. \tag{A15}
$$

Due to the $1/k$ singularity at $k=0$, it is logarithmically divergent as $x\to\infty$. We are interested in its asymptotic behavior. Recall the definition of the cosine integral (here, $\gamma\approx 0.577$ is the Euler–Mascheroni constant)

$$
\mathrm{Ci}(x)=\int_0^x \mathrm{d}q\frac{\cos q-1}{q}+\log x+\gamma \tag{A16}
$$

and its asymptotic form[7]

$$
\mathrm{Ci}(x)\approx\frac{\sin x}{x}+\mathcal{O}(1/x^2)\qquad(x\to\infty). \tag{A17}
$$

Take some $k_*<\infty$ and subtract in the interval $0\leqslant k\leqslant k_*$ the leading term $1/E(k)\approx 1/k$

$$
\begin{aligned}
&\int_0^{k_*}\mathrm{d}k\frac{1-\cos kx}{E(k)}=\log(k_*x)+\gamma-\mathrm{Ci}(k_*x)\\
&-\frac{1}{4}\int_0^{k_*}\mathrm{d}k\frac{k(1-\cos kx)}{\sqrt{1+k^2/4}\,(1+\sqrt{1+k^2/4})}.
\end{aligned}
\tag{A18}
$$

For large x, the cosine integral $\mathrm{Ci}(k_*x)$ vanishes (equation (A17)), and by the Riemann–Lebesgue lemma, the $\cos kx$ can be dropped from the integrand which is regular. The remaining integral is elementary

$$
\begin{aligned}
&-\frac{1}{4}\int_0^{k_*}\mathrm{d}k\frac{k}{\sqrt{1+k^2/4}\,(1+\sqrt{1+k^2/4})}\\
&=-\log\frac{1+\sqrt{1+k_*^2/4}}{2}.
\end{aligned}
\tag{A19}
$$

Consider now the limit $k_*\to\infty$. On the lhs of equation (A18), the integrand scales $\sim 1/k^2$ and falls off sufficiently fast so that one gets the definite integral over $k=0\,\ldots\,\infty$. On the rhs, the integrated term (A19) becomes $-\log(k_*/4)$ so that the logarithms partially compensate. Reinstating the dimensional prefactor, we finally get

$$
|x|\gg\xi:\qquad C_1(x)\approx\frac{\log(2x/\xi)+\gamma}{2\pi\xi}. \tag{A20}
$$

To check this numerically, we keep k_* finite and improve the UV-convergence of the integral over $k_*\leqslant k<\infty$ by adding and subtracting $1/(k^2/2+1)$ under the integral. The added term can be integrated explicitly.

To get the full expression for the correlation function, we recall that the integral (38) also contains the zero-temperature density (depletion). In the limit of large x, by the Riemann–Lebesgue lemma, this piece integrates to equation (35) in the leading order. Combining with equation (A20), we get the result (39).

Appendix B. Details on numerics

We solve implicit equations either with a bisection or an iterative scheme, depending on the convergence rate and *a priori* knowledge about the interval where the solution will be found. In some cases, an interpolation based on parametrically calculated datasets is used. Critical points are determined by minimizing the chemical potential as a

[7] Equation (6.12.3) in Digital Library of Mathematical Functions, `dlmf.nist.gov`.

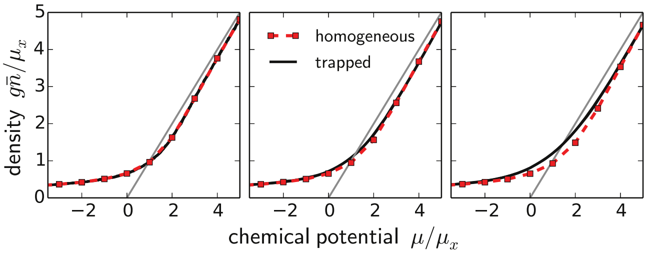

Figure C1. Comparison of the equation of state for a trapped and a homogeneous gas, data from *c*-field simulations with the stochastic Gross–Pitaevskii equation (section 5.1). Red dashed with symbols: spatially averaged density in a homogeneous system; black solid: ensemble-averaged density in a harmonic trap, plotted versus the local chemical potential $\mu = \mu_t - \frac{1}{2}M\omega_z^2 z^2$ and showing only the border of the (quasi-)condensate. $\mu = 0$ corresponds to the Thomas–Fermi radius, the diagonals give the lowest-order Bogoliubov result $\mu = g\bar{n}$. Parameters for Na 23 atoms given in table 2, so that $\beta\mu_x = 0.1$. From left to right, the trap frequency increases through $\hbar\omega_z \approx 9.94, 39.8, 57.2 \times 10^{-3} k_B T$. (Adapted from figure 9.4.2 of [57].)

function of the relevant parameters (e.g., the condensate density, see figure 7).

Appendix C. Validity of the LDA

In figure C1, we compare results obtained with the sGP equation for trapped systems with different trap frequencies. The red (lower) curve is computed for a homogeneous gas (i.e., a sufficiently large box with periodic boundary conditions). The black (upper) curves are based on the LDA and correspond to increasing the axial trapping frequency from left to right. Very good agreement is found on the two asymptotes, but deviations are visible in the cross-over and grow as the trap potential gets steeper. This is consistent with the observation that for the strongest confinement, the inhomogeneity of the potential is significant on the scale of the cross-over: across a displacement of one healing length ξ_x, it changes by a few μ_x.

References

[1] Pitaevskii L P and Stringari S 2003 *Bose–Einstein Condensation, (International Series of Monographs on Physics)* vol 116 (Oxford: Oxford University Press)

[2] Proukakis N P, Gardiner S A, Davis M J and Szymanska M S (ed) 2013 *Quantum Gases: Finite Temperature and Non-Equilibrium Dynamics* (*Cold Atoms* vol 1) (London: Imperial College Press)

[3] Popov V N 1983 *Functional Integrals in Quantum Field Theory and Statistical Physics* (Dordrecht: Reidel) ch 6

[4] Griffin A 1993 *Excitations in a Bose-Condensed Liquid* (*Cambridge Classical Studies*) (Cambridge: Cambridge University Press)

[5] Haldane F D M 1981 Effective harmonic-fluid approach to low-energy properties of one-dimensional quantum fluids *Phys. Rev. Lett.* **47** 1840–3

[6] Proukakis N P and Jackson B 2008 Finite temperature models of Bose–Einstein condensation *J. Phys. B: At. Mol. Opt. Phys.* **41** 203002

[7] Proukakis N P, Morgan S A, Choi S and Burnett K 1998 Comparison of gapless mean-field theories for trapped Bose–Einstein condensates *Phys. Rev.* A **58** 2435–45

[8] Hutchinson D A W, Dodd R J and Burnett K 1998 Gapless finite-T theory of collective modes of a trapped gas *Phys. Rev. Lett.* **81** 2198–200

[9] Griffin A 1996 Conserving and gapless approximations for an inhomogeneous Bose gas at finite temperatures *Phys. Rev.* B **53** 9341–7

[10] Hutchinson D A W, Burnett K, Dodd R J, Morgan S A, Rusch M, Zaremba E, Proukakis N P, Edwards M and Clark C W 2000 Gapless mean-field theory of Bose–Einstein condensates *J. Phys. B: At. Mol. Opt. Phys.* **33** 3825–46

[11] Andersen J O, Al Khawaja U and Stoof H T C 2002 Phase fluctuations in atomic Bose gases *Phys. Rev. Lett.* **88** 070407

[12] Al Khawaja U, Andersen J O, Proukakis N P and Stoof H T C 2002 Low dimensional Bose gases *Phys. Rev.* A **66** 013615
Al Khawaja U, Andersen J O, Proukakis N P and Stoof H T C 2002 *Phys. Rev.* A **66** 059902(E)

[13] Proukakis N P 2006 Spatial correlation functions of one-dimensional Bose gases at equilibrium *Phys. Rev.* A **74** 053617

[14] Walser R, Williams J, Cooper J and Holland M 1999 Quantum kinetic theory for a condensed bosonic gas *Phys. Rev.* A **59** 3878–89

[15] Proukakis N P 2001 Self-consistent quantum kinetics of condensate and non-condensate via a coupled equation of motion formalism *J. Phys. B: At. Mol. Opt. Phys.* **34** 4737–55

[16] Walser R 2004 Ground state correlations in a trapped quasi one-dimensional Bose gas *Opt. Commun.* **243** 107–29

[17] Yukalov V I 2008 Representative statistical ensembles for Bose systems with broken gauge symmetry *Ann. Phys., NY* **323** 461–99

[18] Olivares-Quiroz L and Romero-Rochin V 2010 On the order of bec transition in weakly interacting gases predicted by mean-field theory *J. Phys. B: At. Mol. Opt. Phys.* **43** 205302

[19] Yukalov V I and Yukalova E P 2014 Bose–Einstein condensation in self-consistent mean-field theory *J. Phys. B: At. Mol. Opt. Phys.* **47** 095302

[20] Gardiner C W 1997 Particle-number-conserving Bogoliubov method which demonstrates the validity of the time-dependent Gross–Pitaevskii equation for a highly condensed Bose gas *Phys. Rev.* A **56** 1414–23

[21] Castin Y and Dum R 1998 Low-temperature Bose–Einstein condensates in time-dependent traps: beyond the U(1) symmetry breaking aproach *Phys. Rev.* A **57** 3008–21

[22] Morgan S A 2000 A gapless theory of Bose–Einstein condensation in dilute gases at finite temperature *J. Phys. B: At. Mol. Opt. Phys.* **33** 3847–93

[23] Gardiner S A and Morgan S A 2007 Number-conserving approach to a minimal self-consistent treatment of condensate and noncondensate dynamics in a degenerate Bose gas *Phys. Rev.* A **75** 043621

[24] Billam T P, Mason P and Gardiner S A 2013 Second-order number-conserving description of nonequilibrium dynamics in finite-temperature Bose–Einstein condensates *Phys. Rev.* A **87** 033628

[25] Mora C and Castin Y 2003 Extension of Bogoliubov theory to quasicondensates *Phys. Rev.* A **67** 053615

[26] Blakie P B, Bradley A S, Davis M J, Ballagh R J and Gardiner C W 2008 Dynamics and statistical mechanics of ultra-cold Bose gases using *c*-field techniques *Adv. Phys.* **57** 363–455

[27] Cockburn S P and Proukakis N P 2009 The stochastic Gross–Pitaevskii equation and some applications *Laser Phys.* **19** 558–70
[28] Stoof H T C 1999 Coherent versus incoherent dynamics during Bose–Einstein condensation in atomic gases *J. Low Temp. Phys.* **114** 11–08
[29] Bijlsma M J and Stoof H T C 2001 Dynamics of fluctuating Bose–Einstein condensates *J. Low Temp. Phys.* **124** 431–42
[30] Davis M J, Ballagh R J and Burnett K 2001 Dynamics of thermal Bose fields in the classical limit *J. Phys. B: At. Mol. Opt. Phys.* **34** 4487–512
[31] Gardiner C W, Anglin J R and Fudge T I A 2002 The stochastic Gross–Pitaevskii equation *J. Phys. B: At. Mol. Opt. Phys.* **35** 1555–82
[32] Gardiner C W and Davis M J 2003 The stochastic Gross–Pitaevskii equation: II *J. Phys. B: At. Mol. Opt. Phys.* **36** 4731–53
[33] Góral A, Gajda M and Rzążewski K 2002 Thermodynamics of an interacting trapped Bose–Einstein gas in the classical field approximation *Phys. Rev.* A **66** 051602(R)
[34] Cockburn S P, Gallucci D and Proukakis N P 2011 Quantitative study of quasi-one-dimensional Bose gas experiments via the stochastic Gross–Pitaevskii equation *Phys. Rev.* A **84** 023613
[35] Davis M J, Blakie P B, van Amerongen A H, van Druten N J and Kheruntsyan K V 2012 Yang–Yang thermometry and momentum distribution of a trapped one-dimensional Bose gas *Phys. Rev.* A **85** 031604
[36] Gallucci D, Cockburn S P and Proukakis N P 2012 Phase coherence in quasicondensate experiments: an *ab initio* analysis via the stochastic Gross–Pitaevskii equation *Phys. Rev.* A **86** 013627
[37] Cockburn S P and Proukakis N P 2012 *Ab initio* methods for finite-temperature two-dimensional Bose gases *Phys. Rev.* A **86** 033610
[38] Lieb E H 1963 Exact analysis of an interacting Bose gas: II. The excitation spectrum *Phys. Rev.* **130** 1616–24
[39] Yang C N and Yang C P 1969 Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction *J. Math. Phys.* **10** 1115–22
[40] Kheruntsyan K V, Gangardt D M, Drummond P D and Shlyapnikov G V 2003 Pair correlations in a finite-temperature 1D Bose gas *Phys. Rev. Lett.* **91** 040403
[41] Kheruntsyan K V, Gangardt D M, Drummond P D and Shlyapnikov G V 2005 Finite-temperature correlations and density profiles of an inhomogeneous interacting one-dimensional Bose gas *Phys. Rev.* A **71** 053615
[42] Petrov D S, Shlyapnikov G V and Walraven J T M 2001 Phase-fluctuating 3D Bose–Einstein condensates in elongated traps *Phys. Rev. Lett.* **87** 050404
[43] Hellweg D, Cacciapuoti L, Kottke M, Schulte T, Sengstock K, Ertmer W and Arlt J J 2003 Measurement of the spatial correlation function of phase fluctuating Bose–Einstein condensates *Phys. Rev. Lett.* **91** 010406
[44] Gerbier F, Thywissen J H, Richard S, Hugbart M, Bouyer P and Aspect A 2003 Momentum distribution and correlation function of quasicondensates in elongated traps *Phys. Rev.* A **67** 051602(R)
[45] Al Khawaja U, Proukakis N P, Andersen J O, Romans M W J and Stoof H T C 2003 Dimensional and temperature crossover in trapped Bose gases *Phys. Rev.* A **68** 043603
[46] Gerbier F 2004 Quasi-1D Bose–Einstein condensates in the dimensional crossover regime *Europhys. Lett.* **66** 771–7
[47] Estève J, Trebbia J-B, Schumm T, Aspect A, Westbrook C I and Bouchoule I 2006 Observations of density fluctuations in an elongated Bose gas: ideal gas and quasicondensate regimes *Phys. Rev. Lett.* **96** 130403
[48] Trebbia J-B, Esteve J, Westbrook C I and Bouchoule I 2006 Experimental evidence for the breakdown of a Hartree–Fock approach in a weakly interacting Bose gas *Phys. Rev. Lett.* **97** 250403
[49] Armijo J, Jacqmin T, Kheruntsyan K V and Bouchoule I 2010 Probing three-body correlations in a quantum gas using the measurement of the third moment of density fluctuations *Phys. Rev. Lett.* **105** 230402
[50] Armijo J, Jacqmin T, Kheruntsyan K and Bouchoule I 2011 Mapping out the quasi-condensate transition through the 1D-3D dimensional crossover *Phys. Rev.* A **83** 021605(R)
[51] Bouchoule I, van Druten N J and Westbrook C I 2011 Atom chips and one-dimensional Bose gases *Atom Chips* ed J Reichel and V Vuletić (Weinheim: Wiley) ch 11 pp 331–63
[52] van Amerongen A H, van Es J J P, Wicke P, Kheruntsyan K V and van Druten N J 2008 Yang–Yang thermodynamics on an atom chip *Phys. Rev. Lett.* **100** 090402
[53] Krüger P, Hofferberth S, Mazets I E, Lesanovsky I and Schmiedmayer J 2010 Weakly interacting Bose gas in the one-dimensional limit *Phys. Rev. Lett.* **105** 265302
[54] Yang B, Chen Y-Y, Zheng Y-G, Sun H, Dai H-N, Guan X-W, Yuan Z-S and Pan J-W 2016 Observation of quantum criticality and Luttinger liquid in one-dimensional Bose gases arXiv:1611.00426
[55] Julienne P S 2002 Ultra-cold collisions of atoms and molecules *Scattering–Scattering and Inverse Scattering in Pure and Applied Science* ed R Pike and P Sabatier (London: Academic) ch 2.6.3 pp 1043–67
[56] Weiner J 2003 *Cold and Ultracold Collisions in Quantum Microscopic and Mesoscopic Systems* (Cambridge: Cambridge University Press)
[57] Sauer T-O 2015 Quasi-condensation in low-dimensional Bose gases *Master's Thesis* Universität Potsdam, Potsdam urn: nbn:de:kobv:517-opus4-87247
[58] Lee M D, Morgan S A and Burnett K 2003 The Gross–Pitaevskii equation and higher order theories in one-dimensional Bose gases arXiv:cond-mat/0305416
[59] Astrakharchik G E and Giorgini S 2006 Correlation functions of a Lieb–Liniger Bose gas *J. Phys. B: At. Mol. Opt. Phys.* **39** S1–12
[60] Walls D F and Milburn G J 1994 *Quantum Optics* (Berlin: Springer)
[61] Vogel W, Welsch D-G and Wallentowitz S 2001 *Quantum Optics–An Introduction* (Berlin: Wiley)
[62] Petrov D S, Shlyapnikov G V and Walraven J T M 2000 Regimes of quantum degeneracy in trapped 1D gases *Phys. Rev. Lett.* **85** 3745–9
[63] Pines D and Nozières P 1990 *The Theory of Quantum Liquids, (Advanced Book Classics)* vol 1 (Reading, MA: Perseus)
[64] Silva T D 2014 Evidence for the breakdown of momentum independent many-body t-matrix approximation in the normal phase of bosons *Front. Phys.* **2** 78
[65] Proukakis N P, Burnett K and Stoof H T C 1998 Microscopic treatment of binary interactions in the nonequilibrium dynamics of partially Bose-condensed trapped gases *Phys. Rev.* A **57** 1230–47
[66] Eckart M, Walser R and Schleich W P 2008 Exploring the growth of correlations in a quasi one-dimensional trapped Bose gas *New J. Phys.* **10** 045024
[67] Cockburn S P, Negretti A, Proukakis N P and Henkel C 2011 Comparison between microscopic methods for finite-temperature Bose gases *Phys. Rev.* A **83** 043619
[68] Wright T M, Proukakis N P and Davis M J 2011 Many-body physics in the classical-field description of a degenerate Bose gas *Phys. Rev.* A **84** 023608
[69] Polster H 2015 Das eindimensionale Bose–Einstein-quasi-Kondensat *Diploma Thesis* Universität Potsdam, Germany